Block complexity and idempotent Schur multipliers

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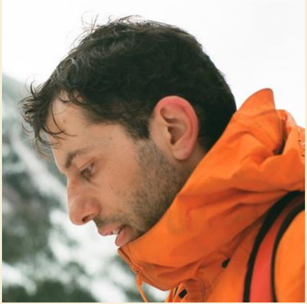
This talk presents joint work with my supervisor Hamed Hatami.



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Theorem (*Cohen*, 1960). *A boolean function $f : \widehat{G} \to \{0, 1\}$ equals $\widehat{\mu}$ for some idempotent measure μ on G if and only if f admits an expression*

$$f = \sum_{i=1}^{L} \pm \mathbf{1}_{s_i + H_i}$$

where L is finite and $s_i + H_i$ are cosets of $\widehat{G}$.

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By Cohen's idempotent theorem, if $\|\widehat{f}\|_1 < \infty$, then we can write $f = \sum_{i=1}^{L} \pm \mathbf{1}_{s_i + H_i}$ for some $L < \infty$ and some cosets $s_1 + H_1, \ldots, s_L + H_L$ of G.

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Proposition (*Kawada–Itô*, 1940). *A nonzero boolean function $f : G \to \{0, 1\}$ satisfies $\|\widehat{f}\| = 1$ if and only if $f = \mathbf{1}_{s+H}$ for some coset $s + H$ of G.*

So if $f = \sum_{i=1}^{L} \pm \mathbf{1}_{s_i + H_i}$, then by the triangle inequality and the above proposition, we have $\|\widehat{f}\|_1 \leq L$. The converse was open for over 50 years...

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Theorem (*Green–Sanders*, 2008). *Let G be a finite abelian group. If $f : G \to \{0,1\}$ satisfies $\|\widehat{f}\|_1 \leq M$, then we can write*

$$f = \sum_{i=1}^{L} \pm \mathbf{1}_{s_i+H_i},$$

where $L \leq \exp(M^{3+o(1)})$ and cosets $s_1 + H_1, \ldots, s_L + H_L$ of G.

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The proof uses tools from modern additive combinatorics: e.g., the Freiman–Ruzsa theorem, the Balog–Szemerédi–Gowers theorem.

We can think of any $n \times n$ matrix A as an operator $A : L_2([n]) \to L_2([n])$ (take $[n] = \mathbf{N}$ if $n = \infty$). The matrix A has an operator norm $\|A\|_{\mathrm{op}}$.

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Each matrix $n \times n$ complex matrix gives rise to a linear transformation on the space of all $A : [n] \times [n] \to \mathbf{C}$ by mapping $A \mapsto M \circ A$, where $\circ$ denotes the Schur (entrywise) product.

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We say that M is a Schur multiplier if and only if it has finite Schur multiplier norm

$$\|M\|_{\mathrm{m}} = \sup_{A \neq 0} \frac{\|M \circ A\|_{\mathrm{op}}}{\|A\|_{\mathrm{op}}}.$$

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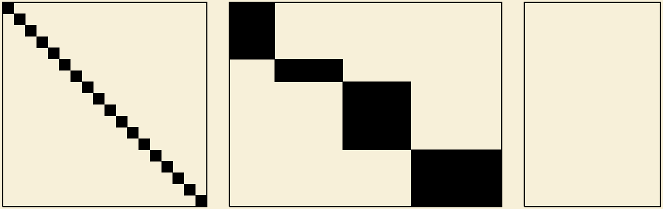
We call a boolean matrix B blocky if there exist families $\{S_i\}_{i\in\mathbf{N}}$ and $\{T_i\}_{i\in\mathbf{N}}$ of pairwise disjoint subsets of $\mathbf{N}$ such that the support of B is exactly $\bigcup_{i\in\mathbf{N}} S_i \times T_i$.

Here are some blocky matrices.



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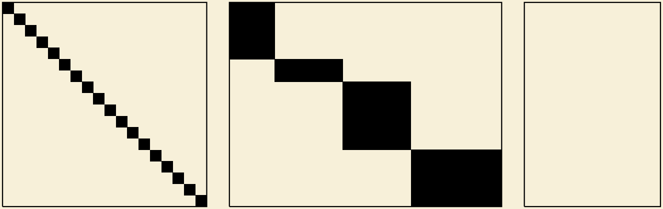

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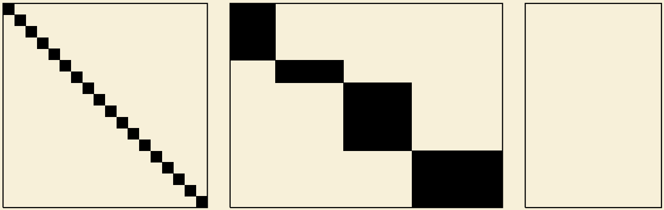

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This is the matrix analogue of the Kawada–Itô result for boolean functions.

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Conjecture H (*Hambardzumyan–Hatami–Hatami*, 2023). *Let A be a finite boolean matrix with $\|A\|_{\mathrm{m}} \leq \gamma$. Then we may express A as a signed sum*

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Given a finite abelian group G and a boolean function $f : G \to \{0, 1\}$, define the matrix $M_f : G \times G \to \{0, 1\}$ by setting $M(x, y) = f(x - y)$. It can be seen that indicators of cosets give rise to blocky matrices, and $\|M_f\|_{\mathrm{m}} = \|\widehat{f}\|_1$, so the Green–Sanders theorem shows that the conclusion of Conjecture H holds for this special class of boolean matrices (convolution matrices).

Theorem (*G.–Hatami*, 2025). *Let A be an $n \times n$ integer matrix with $\|A\|_{\mathrm{m}} \le \gamma$. Then A can be written as a signed sum $A = \sum_{i=1}^{L} \pm B_i$, where each B_i is blocky, and*

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Theorem (*Avraham–Yehudayoff*, 2024). *Let A be an $n \times n$ matrix chosen uniformly at random. Then, with high probability, any decomposition of A as a signed sum of blocky matrices has at least $L \ge n / \big(4 \log_2(2n)\big)$ terms.*

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Caveat. We end up defining A' by averaging certain columns, so we need to deal with real-valued matrices.

Lemma. Let $A \in \mathbf{R}^{X \times Y}$ be a real-valued matrix with $\|A\|_{\mathrm{m}} \leq \gamma$. Suppose further that A is ϵ-almost integer-valued for $\epsilon = 2^{-20\gamma^2}$. If $A_{\mathbf{Z}}$ is not an all-zero matrix, then there exists a 2ϵ-almost integer-valued matrix $A' \in \mathbf{R}^{X \times Y}$ such that

$$\|A - A'\|_{\mathrm{m}}^2 \leq \gamma^2 - \frac{1}{8}$$

and $A' = \sum_{i=1}^{L} \pm B_i$, where the B_i are blocky matrices and

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Here $A_{\mathbf{Z}}$ is the entrywise rounding of A, and we say A is ϵ-almost integer-valued if $\|A - A_{\mathbf{Z}}\|_{\max} \leq \epsilon$.

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