Math 242 Tutorial 9

prepared by

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Problem 1. *Let $A \subseteq \mathbf{R}$. Show that the boundary of A is equal to the boundary of the complement of A.*

Proof. Simply note that

$$\begin{aligned}\partial A &= \bigl\{x \in \mathbf{R} : \text{for all } \epsilon > 0,\ V_\epsilon(x) \cap A \neq \emptyset \text{ and } V_\epsilon(x) \cap A^c \neq \emptyset\bigr\} \\ &= \bigl\{x \in \mathbf{R} : \text{for all } \epsilon > 0,\ V_\epsilon(x) \cap (A^c)^c \neq \emptyset \text{ and } V_\epsilon(x) \cap A^c \neq \emptyset\bigr\} \\ &= \partial(A^c). \quad \blacksquare\end{aligned}$$

Problem 2. *Prove the following:*

a) A set $U \subseteq \mathbf{R}$ is open if and only if U doesn't contain any of its boundary points.

b) A set $A \subseteq \mathbf{R}$ is closed if and only if A contains all of its boundary points.

Proof. Suppose that U contains a boundary point $x \in \partial U \cap U$. Then for this choice of $x \in U$, for any $\epsilon > 0$, $V_\epsilon(x) \cap U^c \neq \emptyset$. This means that $V_\epsilon(x) \not\subseteq U$ for all $\epsilon > 0$. So U is not open. Conversely, if U is not open, there is some $x \in U$ such that for all $\epsilon > 0$, $V_\epsilon(x) \not\subseteq U$, meaning that $V_\epsilon(x) \cap U^c \neq \emptyset$. We also see that $V_\epsilon(x) \cap U$ contains x, so it is also nonempty. This means that x is a boundary point of U; that is, x contains at least one of its boundary points.

By the previous paragraph, A^c is open if and only if A^c doesn't contain any boundary points of A^c, and by the previous paragraph this is true if and only if it doesn't contain any boundary points of A. $\blacksquare$

Problem 3. *Let $\emptyset \neq S \subseteq \mathbf{R}$. Prove the following.*

a) If S is bounded from above, then $\sup S$ is a boundary point of S.

b) If S is bounded from below, then $\inf S$ is a boundary point of S.

Proof. Let $\epsilon > 0$. Since $\sup S - \epsilon$ is not an upper bound of S, there must exist some $s \in S$ with $s > \sup S - \epsilon$. Since $\sup S$ is an upper bound of S, we find that

$$s \in S \cap (\sup S - \epsilon, \sup S] \subseteq S \cap V_\epsilon(\sup S).$$

Also, since $\sup S$ is an upper bound of S, the real number $t = \sup S + \epsilon/2$ is not a member of S. So

$$t \in S^c \cap (\sup S, \sup S + \epsilon) \subseteq S^c \cap V_\epsilon(\sup S).$$

The proof of part (b) is similar. $\blacksquare$

The Cantor set. For any scalar $a \in \mathbf{R}$ and set $S \subseteq \mathbf{R}$, define the *dilate*

$$a \cdot S = \{a \cdot s : s \in S\}$$

as well as the *translate*

$$a + \cdot S = \{a + s : s \in S\}.$$

For integers $n \geq 0$, define C_n recursively by $C_0 = [0, 1]$ and

$$C_{n+1} = \frac{1}{3} \cdot C_n \cup \left(\frac{2}{3} + \frac{1}{3} \cdot C_n\right)$$

for $n \geq 0$. (In other words, C_{n+1} is obtained by removing the "open middle third" from C_n.) The *Cantor set* C is defined to be $C = \bigcap_{n=0}^{\infty} C_n$.

Problem 4. *Prove the following.*

a) *The Cantor set C is closed.*

b) *If $a, b \in \mathbf{R}$ with $a \leq b$ are such that $[a, b] \subseteq C$, then $a = b$. (Hence the only closed intervals in C are singleton sets $\{a\}$.)*

c) *The cardinality of C is equal to that of $[0, 1]$. (That is, C is uncountable.)*

Proof. Note that C_0 is just a closed interval, so *a fortiori* it is a finite union of closed intervals. Now assume that C_n is a finite union of closed intervals. Then $(1/3) \cdot C_n$ is also a finite union of (the same number of) closed intervals, and so is $(2/3) + (1/3)C_n$. Hence C_{n+1} is a finite union of (twice as many) closed intervals. We see by induction that C_n is closed for all $n \geq 0$, and hence $C = \bigcap_{n=0}^{\infty}$ is closed, being an intersection of closed sets.

For part (b), first observe that C_0 is an interval of length $1 = 1/3^0$. Next, assume that C_n is composed of (a finite number of) pairwise disjoint intervals, each of length $1/3^n$. Then C_{n+1} is obtained by taking the disjoint union of two copies of C_n, each dilated by $1/3$, so we conclude that C_{n+1} consists of pairwise disjoint intervals of length $1/3^{n+1}$. By induction, for all integers $n \geq 0$, C_n is composed of disjoint intervals of length $1/3^n$. Now, given $a, b \in \mathbf{R}$ with $a \leq b$. If $a < b$, then $b - a > 0$, so we can pick $N > 1/(b - a)$ by the Archimedean property. Then C_N is composed of disjoint intervals of length $1/3^N < b - a$. We conclude that $[a, b]$ is not a subset of C^N, hence it cannot be contained in C.

Lastly, we show that C is uncountable. Note that every real number $x \in [0, 1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{t_n}{3^n},$$

where $t_n \in \{0, 1, 2\}$ for all $n \in \mathbf{N}$. This is the representation of

$$x = 0.t_1 t_2 t_3 \ldots$$

in base-3 (ternary) notation. Ternary representations aren't unique. For instance, $0.1 = 0.22222\ldots$, just like $0.1 = 0.999999\ldots$ in decimal notation. In ternary notation, $0.022222\ldots = 1/3$ and $0.2 = 2/3$, so the first middle third we removed (in going from C_0 to C_1) contains all numbers of the form $0.1{*}{*}{*}\ldots$, where the ${*}{*}{*}\ldots$ part can be anything strictly between $000\ldots$ and $222\ldots$. That is, C_1 contains all numbers of the form $0.0{*}{*}{*}\ldots$ and $0.2{*}{*}{*}\ldots$, with no restriction whatsoever on the digits ${*}{*}{*}\ldots$.

In going from C_1 to C_2, we remove the middle third of $[0, 1/3]$ as well as the middle third of $[2/3, 1]$. By the same logic we applied above, we see that C_2 contains all numbers of the form $0.00{*}{*}{*}\ldots$, $0.02{*}{*}{*}\ldots$, $0.20{*}{*}{*}\ldots$, or $0.22{*}{*}{*}\ldots$, with no restrictions on the digits ${*}{*}{*}\ldots$. By induction, we find every $x \in C$ can be expressed as

$$x = \sum_{n=1}^{\infty} \frac{t_n}{3^n}$$

where $t_n \in \{0, 2\}$ for all $n \in \mathbf{N}$.

Let $f : C \to [0, 1]$ map

$$x = \sum_{n=1}^{\infty} \frac{t_n}{3^n},$$

where $t_n \in \{0, 2\}$ for all $n \in \mathbf{N}$, to

$$f(x) = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

where we have set

$$b_n = \begin{cases} 0, & \text{if } t_n = 0; \\ 1, & \text{if } t_n = 2. \end{cases}$$

We claim that f is surjective. Every $y \in [0, 1]$ has at least one binary representation

$$y = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

so letting

$$t_n = \begin{cases} 0, & \text{if } b_n = 0; \\ 2, & \text{if } b_n = 1 \end{cases}$$

and

$$x = \sum_{n=1}^{\infty} \frac{t_n}{3^n},$$

we find that $f(x) = y$. The fact that there is a surjective function $f : C \to [0,1]$ means that the cardinality of C is at least that of $[0,1]$. But $C \subseteq [0,1]$, so its cardinality is at most that of $[0,1]$. Hence $|C| = \big|[0,1]\big|$, and we are done. ∎