Math 242 Tutorial 5

prepared by

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9 October 2025

Problem 1. *Prove that*

a) $1/a^n \to 0$ *as* $n \to \infty$*, if* $a > 1$*;*

b) $a^n \to 0$ *as* $n \to \infty$*, if* $0 < a < 1$*; and*

c) $a^n \to 0$ *as* $n \to \infty$*, if* $-1 < a < 0$*.*

Proof. Let $b = a - 1$, so that $b > 0$. Now for $\epsilon > 0$ arbitrary, let $N > 1/(b\epsilon)$. Then for all $n \ge N$,

$$\left|\frac{1}{a^n} - 0\right| = \frac{1}{a^n} = \frac{1}{(1+b)^n} \le \frac{1}{1+nb} < \frac{1}{nb} < \frac{1}{Nb} < \epsilon$$

by Bernoulli's inequality. This shows that $1/a^n \to 0$.

For part (b), let $c = 1/a$. Then $c > 1$ and $a^n = 1/c^n$. It follows from part (a) that

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} \frac{1}{c^n} = 0.$$

For part (c), note that $0 < |a| < 1$, so we know that $|a|^n \to 0$ by part (b). But

$$|a^n - 0| = \big||a|^n - 0\big|$$

for all $n \in \mathbf{N}$, so we see that $a^n \to 0$ as well. ∎

Problem 2. *Prove that*

a) $2^n/n! \to 0$ *as* $n \to \infty$*; and that*

b) $n!/n^n \to 0$*.*

Proof. For part (a), note that $2/k \le 2/3$ for all $k \ge 3$. Hence

$$\frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdots\frac{2}{n} \le \frac{2}{1}\cdot\frac{2}{2}\cdot\left(\frac{2}{3}\right)^{n-2} = 2\cdot\left(\frac{3}{2}\right)^2\left(\frac{2}{3}\right)^n = \frac{9}{2}\cdot\left(\frac{2}{3}\right)^n$$

for all $n \ge 2$. We showed in class that $a^n \to 0$ as $n \to \infty$ whenever $a < 1$. We can apply this with $a = 2/3$ to see that

$$\lim_{n\to\infty} \frac{9}{2}\cdot\left(\frac{2}{3}\right)^n = 0.$$

In other words, for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n \ge N$,

$$\frac{9}{2}\cdot\left(\frac{2}{3}\right)^n < \epsilon.$$

But then we see that

$$0 \le \frac{2^n}{n!} \le \frac{9}{2}\cdot\left(\frac{2}{3}\right)^n < \epsilon,$$

so

$$\left|\frac{2^n}{n!} - 0\right| < \epsilon$$

for all $n \ge N$. Since $\epsilon > 0$ was arbitrary, we are done.

For part (b), we expand

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}.$$

Let $\epsilon > 0$ and pick $N > 1/\epsilon$. Then for all $n \geq N$, we have

$$\left| \frac{n!}{n^n} - 0 \right| = \frac{n!}{n^n} \leq \frac{1}{n} < \epsilon,$$

so we see that $n!/n^n \to 0$. ∎

Problem 3. *Let $a > 1$. Prove that $n/a^n \to 0$ as $n \to \infty$.*

Proof. Let's start by seeing why we can't just do what we did for part (a) of Problem 1 above. Say we set $b = a - 1$, so that $b > 0$. Then for $n \in \mathbf{N}$, we have

$$\left| \frac{n}{a^n} - 0 \right| = \frac{n}{a^n} = \frac{n}{(1+b)^n} \leq \frac{n}{1+nb} < \frac{n}{nb} = \frac{1}{b}$$

by Bernoulli's inequality, but now we're stuck, because $1/b > 0$. Note, however, that we were free to pick b for the purposes of this proof, so we should try to set it to something that creates a larger power of n in the denominator. This will cancel the n in the numerator that is giving us problems.

So let us set $b = \sqrt{a} - 1$. Since $\sqrt{a} > 1$ whenever $a > 1$, we still have $b > 0$. For any $n \in \mathbf{N}$, we have

$$a^n = \left((\sqrt{a})^2\right)^n = (1+b)^{2n} = \left((1+b)^n\right)^2 \geq (1+nb)^2 > n^2 b^2.$$

Now let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ with $N > 1/(\epsilon b^2)$. Then

$$\left| \frac{n}{a^n} - 0 \right| = \frac{n}{a^n} < \frac{n}{n^2 b^2} \leq \frac{1}{nb^2} \leq \frac{1}{Nb^2} < \epsilon$$

for all $n \geq N$. Hence $n/a^n \to 0$. ∎

Problem 4. *Let x_n be a sequence that converges to some limit x. For each $n \in \mathbf{N}$, let a_n be the average*

$$a_n = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Show that a_n converges to x as well.

Proof. Note that

$$|a_n - x| = \left| \frac{1}{n} \sum_{i=1}^{n} x_i - x \right| = \left| \frac{1}{n} \Big(\sum_{i=1}^{n} x_i - nx \Big) \right| = \left| \frac{1}{n} \Big(\sum_{i=1}^{n} (x_i - x) \Big) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |x_i - x|,$$

where in the last line we used the triangle inequality.

Let $\epsilon > 0$ be given. Since x_n converges to x, there exists $N_1 \in \mathbf{N}$ such that for all $n \geq N_1$, $|x_n - x| < \epsilon/2$. This is how we will deal with the terms $|x_i - x|$ for $i \geq N_1$. But we cannot individually bound the terms $|x_i - x|$ when $i < N_1$. But the sum

$$S = \sum_{i=1}^{N_1 - 1} |x_i - x|.$$

is just a (possibly very large) finite number. Let $N_2 \in \mathbf{N}$ be large enough to satisfy $N_2 \geq 2S/\epsilon$. Finally, let $N = \max\{N_1, N_2\}$. Then for any $n \geq N$ we see that

$$\begin{aligned} |a_n - x| &\leq \frac{1}{n} \sum_{i=1}^{N_1 - 1} |x_i - x| + \frac{1}{n} \sum_{i=N_1}^{n} |x_i - x| \\ &\leq \frac{S}{n} + \frac{1}{n} \sum_{i=N_1}^{n} \frac{\epsilon}{2} \\ &\leq \frac{S}{N_2} + \frac{n - N_1 + 1}{n} \cdot \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This shows that $a_n \to x$ as $n \to \infty$. ▮