Math 242 Tutorial 2

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Problem 1. *Let $f : D \to E$ be a function and let $B \subseteq E$. Prove that*

a) $f\big(f^{-1}(B)\big) \subseteq B$*; and that*

b) if f is surjective, then $f\big(f^{-1}(B)\big) = B$.

Proof. Let $y \in f\big(f^{-1}(B)\big)$. There is some $x \in f^{-1}(B)$ such that $f(x) = y$. Since $x \in f^{-1}(B)$, we must have $f(x) \in B$. But we have $y = f(x)$, so in fact $y \in B$. This settles part (a).

For part (b), assume that f is surjective. In view of part (a) we only have to prove that $B \subseteq f\big(f^{-1}(B)\big)$. Let $y \in B$; since f is surjective, there is some $x \in D$ such that $f(x) = y$. This x, then, is a member of $f^{-1}(B)$ (being an element that is mapped to y, which we assumed to be in B). But then we see that $y \in f\big(f^{-1}(B)\big)$, since $y = f(x)$. ∎

Problem 2. *Find the supremum and infimum of each of the following subsets of* $\mathbf{R}$ *(and prove that each is a supremum/infinum).*

a) $A = \big\{1/2^n : n \in \mathbf{N}\big\}$.

b) $B = \big\{(-1)^n + 1/n : n \in \mathbf{N}\big\}$.

Proof. For part (a), note that if $m > n$, then $1/2^m < 1/2^n$, so the greatest element of A, namely $1/2$, is also its supremum. The infimum, on the other hand, is 0, which we now prove. First we note that $0 \le 1/2^n$ for all $n \in \mathbf{N}$. Then, let any $\epsilon > 0$ be given. We need to find some n such that $1/2^n < \epsilon$. But this can be done, but the Archimedean property of $\mathbf{R}$. Let n be a natural number such that $n > \log_2(1/\epsilon)$. Then

$$\frac{1}{2^n} < \frac{1}{2^{\log_2(1/\epsilon)}} = \frac{1}{1/\epsilon} = \epsilon,$$

and we are done.

For part (b), it helps to write out some elements of B for small $n \in \mathbf{N}$. We have

$$-1 + 1/1 = 0, \qquad 1 + 1/2 = 3/2, \qquad -1 + 1/3 = -2/3, \qquad 1 + 1/4 = 5/4, \qquad -1 + 1/5 = -4/5,$$

and so on. We see that each even n gives an element of B that is larger than 1, but as n gets larger, these elements will get closer and closer to 1. For odd n, we get an element in the interval $(-1, 0]$, but again, as n gets larger, the $1/n$ term gets smaller so it these elements get closer and closer to -1. We claim, then, that $\sup B = 3/2$, and $\inf B = -1$.

First we prove the claim about the supremum. For all odd n, the number $(-1)^n + 1/n$ equals $-1 + 1/n$, which is nonpositive, so $3/2 > (-1)^n + 1/n$. For all even n, we have $(-1)^n + 1/n = 1 + 1/n$, which is at most $3/2$. So $3/2$ is an upper bound for the set, The element $3/2$ is in the set, so no number that is smaller than $3/2$ can be an upper bound on B. Now we prove the claim about the infimum. It is easy to check that -1 is a lower bound on B, since all even n give a positive number, and all odd n give $-1 + 1/n \ge -1$. We just have to show now that it is the *greatest* lower bound. Let $\epsilon > 0$. We need to find some odd n such that $-1 + 1/n < -1 + \epsilon$. To this end, we use the Archimedean property of $\mathbf{R}$ again, picking some n such that $n > 1/\epsilon$. Without loss of generality, we can choose n odd (since if n is even, then $n + 1$ is odd and is also greater than $1/\epsilon$). Then we see that

$$-1 + \frac{1}{n} < -1 + \frac{1}{1/\epsilon} = -1 + \epsilon,$$

and we are done. ∎

Problem 3. *Prove that for all $x \in \mathbf{R}$,*

a) $|x| = |-x|$;

b) $|x| = \sqrt{x^2}$;

c) $-|x| \le x \le |x|$;

and for all $x, y \in \mathbf{R}$,

d) $|xy| = |x| \cdot |y|$.

Proof. If x is nonnegative, then $-x$ is nonpositive, so $|x| = x = |-x|$. If x is negative, then $-x$ is positive, so $|x| = -x = |-x|$. Either way, $|x| = |-x|$, and part (a) is done.

For any real number a, the square root $\sqrt{a}$ is defined to be the unique nonnegative real number b with $b^2 = a$. Note that $|x|^2 = x^2$; this is because if x is nonnegative, then $|x| = x$ and the identity is clear, and if x is negative, then $|x| = -x$ and we have $|x|^2 = (-x)^2 = (-1)^2x^2 = x^2$. But $|x|$ is nonnegative by construction, and satisfies $|x|^2 = x^2$, so we conclude that $|x| = \sqrt{x^2}$.

For part (c), note that if x is negative, then $|x| = -x$, so $-|x| \le x < -x = |x|$, and if x is nonnegative, then $|x| = x$, so $-|x| = -x \le x = |x|$.

Equipped with the identity from part (b), we have

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2} = \sqrt{x^2}\sqrt{y^2} = |x| \cdot |y|,$$

since multiplication of real numbers is commutative. This proves part (d). ∎

Problem 4. *Prove the reverse triangle inequality*

$$|x - y| \ge \big||x| - |y|\big|,$$

which holds for all $x, y \in \mathbf{R}$.

Proof. Note first that

$$(x - y)^2 = x^2 - 2xy + y^2.$$

The terms x^2 and y^2 are both nonnegative, but the middle term, $-2xy$, could be positive or negative, depending on the relative signs of x and y. If we change this term to $-2|x| \cdot |y|$, then the term must be negative, and the whole right-hand side either stays the same or goes down; in other words,

$$(x - y)^2 \ge x^2 - 2|x| \cdot |y| + y^2.$$

But by part (c) of the previous problem, we see that

$$(x - y)^2 \ge |x|^2 - 2|x| \cdot |y| + |y|^2 = \big(|x| - |y|\big)^2.$$

Taking square roots of both sides (and applying part (c) of the previous problem once again), we have

$$|x - y| \ge \big||x| - |y|\big|,$$

which is what we wanted. ∎

Here's another proof using the ordinary triangle inequality that was proved in class.

Alternative proof. We add the "clever" zero $-y + y$ to x, obtaining

$$|x| = \big|(x - y) + y\big| \le |x - y| + |y|$$

from the triangle inequality. This rearranges to $|x - y| \ge |x| - |y|$. Likewise, we have the similar bound

$$|y| = \big|(y - x) + x\big| \le |y - x| + |x| = |x - y| + |x|,$$

where after the triangle inequality we used the identity from part (a) of the previous question. This rearranges to $|x - y| \geq |y| - |x|$, so we have shown that $|x - y| \geq \big||x| - |y|\big|$. ∎

Problem 5. *Let $a, b \in \mathbf{R}$. Show that $a = b$ if and only if for all $\epsilon > 0$, we have $|a - b| \leq \epsilon$.*

Proof. The "only if" direction is easy. Assume that $a = b$ and let $\epsilon > 0$ be arbitrary. We have

$$|a - b| = |a - a| = 0 \leq \epsilon.$$

Now we show the "if" direction, by contrapositive. (We must show that $a \neq b$ implies the existence of some $\epsilon > 0$ such that $|a - b| > \epsilon$.) Suppose that $a \neq b$, so that $a - b \neq 0$. This means that $|a - b| > 0$, so we can set $\epsilon = |a - b|/2$, which is also positive. Then we have

$$|a - b| > \frac{|a - b|}{2} = \epsilon,$$

which settles the proof. ∎