Math 242 Tutorial 11

prepared by

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Problem 1. *Let $D \subseteq \mathbf{R}$ and let $f, g : D \to \mathbf{R}$ be functions that are continuous on all of D.*

a) *Show that the function $f - g$ is continuous on all of D.*

b) *Suppose further that for all $x \in D$ we also have $-x \in D$. (So D is symmetric about the origin.) Let $h : D \to \mathbf{R}$ be given by $h(x) = f(-x)$. Show that h is continuous on all of D.*

Proof. Let $p \in D$. Since f and g are both continuous at p, we have $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to p} g(x) = g(p)$. Hence

$$\lim_{x\to p}(f-g)(x) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x) = f(p) - g(p),$$

by the algebraic laws for limits of functions.

Now we prove part (b). Let $\epsilon > 0$ and let $p \in D$. Since $-p \in D$ and f is continuous on all of D, f is continuous at $-p$. Hence there exists some $\delta > 0$ such that for all $y \in D$ with $|y - p| < \delta$, one has $\bigl|f(y) - f(-p)\bigr| < \epsilon$. Let $x \in D$ be such that $|x - p| < \delta$. Then

$$\bigl|-x-(-p)\bigr| = \bigl|-(x-p)\bigr| = |x-p| < \delta,$$

so letting $y = -x$ above gives tells us that

$$\bigl|h(x) - h(p)\bigr| = \bigl|f(-x) - f(-p)\bigr| < \epsilon.$$

This shows that h is continuous at p, and since p was selected arbitrarily, we conclude that h is continuous on all of D. ∎

These statements generalise to $D \subseteq \mathbf{R}^n$, a fact you may use in the following problem.

Problem 2. *Let S^2 denote a two-dimensional sphere in $\mathbf{R}^3$, which can be shifted and scaled so that*

$$S^2 = \bigl\{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\bigr\}.$$

Let $f : S^2 \to \mathbf{R}$ be continuous. Using the location of roots theorem, show that there exists a point $p \in S^2$ such that $f(-p) = f(p)$.

Proof. The set S^2, as we have defined it above, is symmetric about the origin of $\mathbf{R}^3$, so by the previous problem, the function $g : S^2 \to \mathbf{R}$ given by $g(v) = f(v) - f(-v)$ is continuous. Now fix any point $v \in S^2$ and consider any semicircular arc in S^2 from v to $-v$. Let $\gamma : [0,1] \to \mathbf{R}^3$ parameterise this arc, so that $\gamma(0) = v$ and $\gamma(1) = -v$ (and say, $\gamma(1/3)$ is one-third of the way from v to $-v$ along this arc). This is a continuous function (this technically needs to be proved, but for the purposes of this class, let's just take it as fact).

Let $h : [0,1] \to \mathbf{R}$ be given by $h(t) = g\bigl(\gamma(t)\bigr)$. This is continuous, since it is the composition of continuous functions. First, note that

$$h(0) = g\bigl(\gamma(0)\bigr) = f(v) - f(-v) = -\bigl(f(-v) - f(v)\bigr) = -g\bigl(\gamma(1)\bigr).$$

If $h(0) = 0$, then $g\bigl(\gamma(0)\bigr) = 0$, meaning that $g(v) = 0$; that is, $f(v) = f(-v)$, so we can set $p = v$ and we are done. If $h(0) \neq 0$, then $h(1) \neq 0$ also, and they have opposite signs. Hence by the location of roots theorem, there is some $t \in (0,1)$ with $h(p) = 0$. Let $p = \gamma(t)$; we have

$$0 = g\bigl(\gamma(t)\bigr) = f(p) - f(-p),$$

which means that $f(p) = f(-p)$, which is what we wanted. ∎

The previous problem shows that if we assume air temperature to be a continuous function on the globe, then there is some point on Earth that has the same temperature as its antipode (the opposite point on the globe). In fact, using a stronger topological theorem called the *Borsuk–Ulam theorem*, one can show that the same thing holds for *pairs* of continuous functions on the sphere. For instance, if we assume barometric pressure to be continuous on the sphere as well, then there is a point on Earth with the same air temperature *and* the same barometric pressure as its antipode.

Problem 3. *Prove that* $\mathbf{R}$ *has the least upper bound property using the localization of roots theorem (and without using anything else that we have shown to be equivalent to the completeness axiom).*

Proof. Suppose, for a contradiction, that there is some $S \subseteq \mathbf{R}$ that is nonempty and does not have a least upper bound. We define a function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } f \text{ is an upper bound of } S; \\ -1, & \text{if } f \text{ is not an upper bound of } S. \end{cases}$$

First we show that f is continuous at every point $c \in \mathbf{R}$. To do this, let $c \in \mathbf{R}$ and let $\epsilon > 0$. There are two cases.

If $f(c) = 1$, then c is an upper bound of S. But S does not have a least upper bound, so there is some $b < c$ such that b is also an upper bound of c. In this case, set $\delta = c - b > 0$. For all x with $|x - c| < \delta$, we have $x > b$, so x is an upper bound of S and $f(x) = 1$. We see that

$$\big|f(x) - f(c)\big| = |1 - 1| = 0 < \epsilon.$$

If, on the other hand, $f(c) = -1$, then c is not an upper bound of S, and there is some $s \in S$ with $s > c$. In this case, set $\delta = s - c > 0$. Then for all x with $|x - c| < \delta$, we must have $x < s$, so x is not an upper bound of S and $f(x) = 0$. Hence

$$\big|f(x) - f(c)\big| = \big|-1 - (-1)\big| = 0 < \epsilon.$$

We have shown that f is continuous on all of $\mathbf{R}$.

Now let $s \in S$ and let t be any upper bound of S (so we must have $s \leq t$). The real number $s - 1$ is not an upper bound of S, so $f(s - 1) = -1$. On the other hand, $f(t) = 1$. So by the location of roots theorem, there must be some $x \in [s - 1, t]$ with $f(x) = 0$. But this is a contradiction, since $f(x) \in \{-1, 1\}$ for all $x \in \mathbf{R}$. ∎

Problem 4. *Show that a set* $f : \mathbf{R} \to \mathbf{R}$ *is continuous on all of* $\mathbf{R}$ *if and only if for every open set* $U \subseteq \mathbf{R}$*, the inverse image* $f^{-1}(U)$ *is open.*

Proof. Suppose that f is continuous and let U be an open set. We want to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, so that $f(x) \in U$. Since U is open, there exists some $\epsilon > 0$ such that $V_\epsilon\big(f(x)\big) \subseteq U$. But using this ϵ in the definition of continuity of f, there must be some $\delta > 0$ such that for all $y \in V_\delta(x)$, $f(y)$ is in $V_\epsilon\big(f(x)\big)$, which is a subset of U. But this implies that $V_\delta(x) \subseteq f^{-1}U$. Since x was arbitrary, we have shown that $f^{-1}(U)$ is open.

Now suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function such that for every open set $U \subseteq \mathbf{R}$ the set $f^{-1}(U)$ is also open. Let $x \in \mathbf{R}$ and $\epsilon > 0$. The set $U = V_\epsilon\big(f(x)\big)$ is open, so by hypothesis its inverse image $f^{-1}(U)$ is open. Of course, x is in $f^{-1}(U)$, so there is some $\delta > 0$ such that $V_\delta(x) \subseteq f^{-1}(U)$. Unpacking the definitions of $V_\delta(x)$ and $U = V_\epsilon\big(f(x)\big)$, we see that for this choice of δ, we have $\big|f(x) - f(z)\big| < \epsilon$ whenever $|x - z| < \delta$. Hence f is continuous. ∎

This problem justifies the definition of continuity over $\mathbf{R}$ that is used in the class, since over general topological spaces a function is *defined* to be continuous if the inverse image of every open set is open.

Problem 5. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuous function and let* $p \in \mathbf{R}$*. Show that the set*

$$S = \big\{x \in \mathbf{R} : f(x) = p\big\}$$

is closed.

Proof. The set $\{p\} = [p, p]$ is closed, so its complement $U = \mathbf{R} \setminus \{p\}$ is open. Note that $\mathbf{R} \setminus S = f^{-1}(U)$, so by the previous problem, $\mathbf{R} \setminus S$ is open. Hence S is closed. ∎