Math 242 Tutorial 10

prepared by

Marcel Goh

20 November 2025

Problem 1. *Let $p \in \mathbf{R}$ be a point contained in some interval I. Suppose that f and g are functions defined on $I \setminus \{p\}$ with $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = M$ for some $L, M \in \mathbf{R}$. Prove that*

a) $\lim_{x\to p} C \cdot f(x) = CL$ *for all* $C \in \mathbf{R}$*;*

b) $\bigl(\lim_{x\to p} f(x)\bigr)\bigl(\lim_{x\to p} g(x)\bigr) = LM$*; and*

c) $\bigl(\lim_{x\to p} f(x)\bigr)\big/\bigl(\lim_{x\to p} g(x)\bigr) = L/M$*, so long as* $M \ne 0$ *and* $g(x) \ne 0$ *for all* $x \in I \setminus \{p\}$*.*

Proof. Let x_n be a sequence in $I \setminus \{p\}$ that converges to p. Then $f(x_n) \to L$ as $n \to \infty$, and $g(x_n) \to M$ as $n \to \infty$. This whole question can now be solved using the laws that govern limits of sequences.

For part (a), we use the fact that

$$\lim_{n\to\infty} c \cdot f(x_n) = cL.$$

For part (b), observe that

$$\Bigl(\lim_{n\to\infty} f(x_n)\Bigr)\Bigl(\lim_{n\to\infty} g(x_n)\Bigr) = \lim_{n\to\infty} f(x_n)g(x_n) = LM.$$

Lastly, for part (c) we have

$$\lim_{n\to\infty} f(x_n)\Big/\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{L}{M}.$$

All three parts now follow from the sequential definition of the limit, as well as the fact that x_n was taken to be an arbitrary sequence. ∎

Problem 2. *Let $f : \mathbf{R} \to \mathbf{R}$ be defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}; \\ 0, & \text{if } x \in \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

(This function is called the Dirichlet function.) Show that for any $c \in \mathbf{R}$, the limit $\lim_{x\to c} f(x)$ does not exist.

Proof. Let $c \in \mathbf{R}$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$, for any $n \in \mathbf{N}$ there are infinitely many rational numbers in the ball $V_{1/n}(c)$. Hence there are also infinitely many rational numbers in the punctured ball $V^*_{1/n}(c)$. Hence we may construct a sequence (r_n) in $\mathbf{Q} \setminus \{c\}$ with $|r_n - c| < 1/n$ for all $n \in \mathbf{N}$. This sequence converges to c.

But $\mathbf{R} \setminus \mathbf{Q}$ is also dense in $\mathbf{R}$, so by the same logic as in the previous paragraph there is a sequence (s_n) in $(\mathbf{R} \setminus \mathbf{Q}) \setminus \{c\}$ that converges to c. Now observe that $\lim_{n\to\infty} f(r_n) = 1$ and $\lim_{n\to\infty} f(s_n) = 0$, so $f(x)$ cannot have any limit at $x = c$. ∎

Problem 3. *Let $f : \mathbf{R} \to \mathbf{R}$ be defined by*

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \in \mathbf{Q} \text{ with } p \in \mathbf{Z},\ q \in \mathbf{N}, \text{ and } \gcd(p,q) = 1; \\ 0, & \text{if } x \in \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

(This function is called Thomae's function, or the modified Dirichlet function, or the stars over Babylon.)

a) Show that f is periodic with period 1; that is, $f(x+n) = f(x)$ for all integers n and all $x \in \mathbf{R}$.

b) Show that for any $c \in \mathbf{Q}$, we have $\lim_{x\to c} f(x) \ne f(c)$.

c) Show that for any $s \in \mathbf{R} \setminus \mathbf{Q}$ one has $\lim_{x\to s} f(x) = f(s)$.

Proof. Recall that the sum of an irrational number and a rational number is irrational, and that the quotient of an irrational number by a rational number is also irrational.

Let $n \in \mathbf{Z}$ and $x \in \mathbf{R}$. If x is irrational, then $x + n$ is also irrational, and $f(x+n) = 0 = f(x)$. If x is rational, we may express $x = p/q$ with $p \in \mathbf{Z}$, $q \in \mathbf{N}$, and $\gcd(p,q) = 1$, so that $f(x) = 1/q$. Then

$$x + n = \frac{p}{q} + n = \frac{p+nq}{q}.$$

If we can show that $\gcd(p+nq, q) = 1$, then $f(x+n) = q$ and we are done. Suppose that d divides both p and q; say, $p = rd$ and $q = sd$ for some integers r and s. Then

$$p + nq = rd + nsd = (r+ns)d,$$

so d divides $p + nq$ as well. On the other hand, if d divides both $p + nq$ and q; say $p + nq = rd$ and $q = sd$ for some $r, s \in \mathbf{Z}$. Then

$$p = (p+nq) - nq = rd - nsd = (r-ns)d,$$

so d divides p as well. We have shown that the common divisors of p and q are exactly the common divisors of $p + nq$ and q. So $\gcd(p+nq, q) = \gcd(p,q) = 1$.

Let $c \in \mathbf{Q}$ be arbitrary and express $c = p/q$, where p and q are integers with $q > 0$ and $\gcd(p,q) = 1$. We have $f(c) = 1/q$ by the definition of f. The claim is that $\lim_{x\to c} f(x) \neq 1/q$. So we must show that there exists an ϵ such that for all $\delta > 0$, there exists $x \in \mathbf{R} \setminus \{c\}$ with $|x - c| < \delta$ and $\big|f(x) - 1/q\big| \geq \epsilon$. Fix any positive irrational number α. We pick $\epsilon = 1/q$ and let $\delta > 0$ be arbitrary. Using the Archimedean property, choose $n \in \mathbf{N}$ with $n > \alpha/\delta$ (so that $\alpha/n < \delta$), and set

$$x = c + \frac{\alpha}{n}.$$

From the observation in the first paragraph of this proof, we see that x is irrational, so $f(x) = 0$ and $\big|f(x) - 1/q\big| = 1/q \geq \epsilon$. On the other hand, we have

$$|x - c| = \left|\frac{\alpha}{n}\right| < \delta,$$

so we conclude that the limit $\lim_{x\to c} f(x)$ does not exist.

Now let s be irrational, so that $f(s) = 0$. The claim is that $\lim_{x\to s} f(x) = 0$. Observe that $s = t + n$ for some integer n and some $t \in (0,1)$, and if $\lim_{x\to t} f(x) = 0$ then

$$\lim_{x\to s} f(x) = \lim_{x\to t} f(x+n) = \lim_{x\to t} f(x) = L$$

as well. So to show that $\lim_{x\to s} f(x) = 0$, it suffices to show that $\lim_{x\to t} f(x)$. Let $\epsilon > 0$ and pick $m \in \mathbf{N}$ with $1/m < \epsilon$ using the Archimedean property. For each $i \in \{1, \ldots, m\}$, let k_i be the integer with

$$0 < \frac{k_i}{i} < t < \frac{k_i + 1}{i}.$$

For all $1 \leq i \leq m$, let d_i be the minimal distance between s and either k_i/i or $(k_i+1)/i$; that is,

$$d_i = \min\left\{\left|t - \frac{k_i}{i}\right|, \left|t - \frac{k_i+1}{i}\right|\right\}.$$

Note that $d_i > 0$ for all $1 \leq i \leq m$, so if we set

$$\delta = \{d_1, \ldots, d_m\},$$

then $\delta > 0$, and for all $1 \leq i \leq m$, $|t - k_i/i| \geq \delta$ and $\left|t - (k_i + 1)/i\right| \geq \delta$. In other words, for all $1 \leq i \leq m$, both of the rational numbers k_i/i and $(k_i + 1)/i$ are outside the ball $V_\delta(s)$. What this means is that any rational number in $V_\delta(t)$ must have a denominator greater than m. Hence for any $x \in \mathbf{R}$ with $|x - t| < \delta$, either x is irrational, in which case

$$\left|f(x) - 0\right| = |0 - 0| = 0 < \epsilon,$$

or x is rational, in which case x can be written as $x = p/q$ with $p \in \mathbf{Z}$, $q > m$, and $\gcd(p, q) = 1$. In this second case,

$$\left|f(x) - 0\right| = \left|\frac{1}{q} - 0\right| = \frac{1}{q} < \frac{1}{m} \leq \epsilon,$$

and we have shown that $\lim_{x \to t} f(x) = 0$, as desired. ∎