Math 242 Tutorial 1

prepared by

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Problem 1. *Let p and q be statements. Show that $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$.*

Proof. We have

$$\begin{aligned} p \Rightarrow q &\equiv \neg p \vee q \\ &\equiv q \vee \neg p \\ &\equiv \neg(\neg q) \vee \neg p \\ &\equiv \neg q \Rightarrow \neg p \quad \blacksquare \end{aligned}$$

Problem 2. *Let n be an integer. Show that if n is not a perfect square, then $\sqrt{n}$ is irrational.*

Proof. By contrapositive. Suppose $\sqrt{n}$ is rational. Then there exist $a, b \in \mathbf{Z}$ (with $b \neq 0$) such that $\sqrt{n} = a/b$. Without loss of generality, we can assume that $\gcd(a, b) = 1$. We see that $n = a^2/b^2$, but since n is an integer, and a^2 and b^2 have no common factors, we must have $b^2 = 1$. This forces $n = a^2$, which shows that n is a perfect square. $\blacksquare$

Problem 3. *Prove the identity*

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for all integers $n \geq 1$.

Proof. By induction on n. For the base case $n = 1$, we have

$$\sum_{k=1}^{1} k^3 = 1^3 = 1^2 = \left(\frac{1 \cdot 2}{2}\right)^2 = \left(\frac{1 \cdot (1+1)}{2}\right)^2.$$

Now suppose the statement holds for some n. For $n + 1$ we have

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= (n+1)^3 + \sum_{k=1}^{n} k^3 \\ &= (n+1)^3 + \left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{4(n+1)(n+1)^2 + n^2(n+1)^2}{4} \\ &= \frac{(n^2+4n+4)^2(n+1)^2}{4} \\ &= \left(\frac{(n+1)\big((n+1)+1\big)}{2}\right)^2, \end{aligned}$$

where in the second line, we used the induction hypothesis. $\blacksquare$

Problem 4. *Suppose that a local McDonalds branch sells Chicken McNuggets in meals of 4 or 9. Prove that for all $n \geq 36$, it is possible to buy exactly n Chicken McNuggets.*

Proof. Formulated precisely in mathematical language, what we are trying to show is that for every integer $n \geq 36$, there exist nonnegative integers a and b such that $n = 9a + 4b$. We perform a proof by strong induction. Note first that

$$36 = 9 \cdot 4 + 4 \cdot 0, \qquad 37 = 9 \cdot 1 + 4 \cdot 7, \qquad 38 = 9 \cdot 2 + 4 \cdot 5,$$

and

$$39 = 9 \cdot 3 + 4 \cdot 3.$$

Now let $n \geq 40$ and assume that the statement holds for all integers in the range $[36, n]$. In particular, it holds for $n - 4$. So there exist nonnegative integers a and b such that $n - 4 = 9a + 4b$, and we see that $n = 9a + 4(b + 1)$. ∎

Problem 5. *Show that there exist irrational numbers a and b such that a^b is rational.*

Proof. Consider the number $(\sqrt{2})^{\sqrt{2}}$. It is either rational or irrational.

If it is rational, then we can set $a = b = \sqrt{2}$, which we already showed to be irrational in class, and a^b is rational by the assumption in this case.

If it is irrational, then we can set $a = (\sqrt{2})^{\sqrt{2}}$ and $b = \sqrt{2}$. We compute

$$a^b = \left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2,$$

which is rational. ∎