MATH 140 Fall 2026

notes by

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A note on these notes. This document will periodically be updated with the material that we cover. The datestamps in the left margin indicate when the notes from each day start. Subsections labelled with a $*$ are optional. These notes should *not* be used as a substitute for the textbook or for attending class. (For instance, on the blackboard I will often draw graphs or diagrams, and these for the most part will be omitted from the notes.) The best way to do well in this class is to do as many exercises from Stewart's textbook as you can stomach. This document may contain errors, typographical or mathematical. Please email me if you find any.

I. FOUNDATIONS

Que si cependant la Geometrie a toûjours
quelque obscurité essentiel, qu'on ne puisse dissiper,
& ce fera uniquement, à ce que je crois, du côté de l'Infini,
c'est que de ce côté-là la Geometrie tient à la Phisique,
à la nature intime des Corps que nous connoissons peu,
& peut-être aussi à une Metaphisique trop élevée,
dont il ne nous est permis que d'appercevoir quelques rayons.

— BERNARD LE BOVIER DE FONTENELLE, *Élements de la Geometrie de l'Infini* (1727)

1. Rates of change

31.VIII This set of notes assumes that the reader is familiar with all the pre-calculus concepts that one learns in the later years of secondary school: functions, trigonometry, exponential functions, logarithms, etc. However, given that the very definition of what differential calculus is requires us to talk about functions and their inputs, it is perhaps prudent to begin with a refresher on real-valued functions with real-valued outputs.

A *function* is a collection of ordered pairs (x, y) with the rule that if (x, y) and (x, z) are both in the collection, then $y = z$. We can view x as the *input* to the function, and y as the *output* of the function; under this view, the rule above is simply that the function never outputs two different values when given the same input. People like to use the letter f (or g or h, but really we can use any letter) to denote a function, and $f(x)$ to denote its value evaluated at a given input x. However, many particular functions are difficult to write without involving the input variable x in the expression, and in these cases we'll write $f(x)$ to mean the function as well.

Here are some examples of functions:

a) The function $f(x) = 3x$, defined on all real numbers, that outputs triple its input value.
b) The function that assigns to each real number x its square x^2.
c) The function $y = f(x)$ that equals 0 whenever x is a negative real number, and equals x whenever $x \geq 0$.
d) The function that takes as input any nonzero real number x and outputs its reciprocal $1/x$.
e) The function $f(x)$, defined on all real numbers, given by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Note that in each case above, we were careful to specify what the possible inputs to the function were. Going back to the definition of a function as a collection of ordered pairs (x, y), the set of all x such that (x, y) is in the function is called the *domain* of f; this is precisely the set of all valid inputs to the function f. (The set of all y such that (x, y) is in f is called the *range* of f, but this definition won't be as important to us.)

The title of this section is "rates of change," but determining the rate at which a given function changes is actually what this *entire class* will essentially be about. We want to study how much the output of a function changes when you change its input.

For example, consider a car moving from Montreal to Quebec City. As time goes on, it gets further away from Montreal and closer to Quebec City. So we can model the trip as a function $d(t)$, where for each positive real number t we let $d(t)$ equal the distance of the car from Montreal (in kilometres), exactly t

minutes since it set off. The rate of change of this function at any given point in time t is the car's *velocity* at that point.

How do we calculate velocity? Well, we know that it is measured in units such as "kilometres per hour" or "metres per second," so it is not too hard to see that velocity is calculated by dividing distance over time.

Let's go back to the car example. Suppose your car is equipped with a very good GPS that can measure your distance from Montreal; however, it does not have a working speedometer. Your friend (who is driving) asks you, "How fast am I driving?" You have taken high school physics, so you know how to solve this problem. You get out a stopwatch and start it exactly when the GPS measures your distance at 135.2 kilometres from Montreal. After one minute on the stopwatch, the GPS reads 137.4, so you report to your friend that she is driving $137.4 - 135.2 = 2.2$ kilometres per minute, or 132 kilometres per hour.

But your friend is not satisfied. "You told me my average speed over the course of a minute, not my speed at the moment I asked you the question. During that minute, I had to speed up to overtake a U-Haul and then slow down because someone cut me off in the fast lane."

So you try and do better. You get out the stopwatch again, and when you start it the GPS reads 137.7. You wait ten seconds, then observe that the GPS reads 138.1. So you report your friend's new speed at $138.1 - 137.7 = 0.4$ kilometres every ten seconds, or 2.4 kilometres per minute, or 144 kilometres per hour.

Your friend is a little more satisfied, but wonders if you can do better. So you try one last time. You start the stopwatch when the GPS reads 138.5, wait exactly one second, and then check the GPS again. To everyone's surprise and disappointment, it still reads 138.5, because it only measures distance to the tenth part of a kilometre. So the calculation would report that the car has moved 0 kilometres in that one second, that is, it's going 0 kilometres per hour.

Obviously in this example you were hamstrung by the precision of your measuring instruments, but it is still worth pondering the following point. Your friend was asking for her speed at some particular instant. However, during any particular instant she does not move, nor does time move. So any naïve calculation as to her speed would go along the lines of $0/0$, which is an undefined value.

Yet it is undeniable that something's "speed at a particular instant" is a concept that holds meaning in everyday life. If you are in a plane and your friend is in a car below you, then it's certainly not nonsensical to say that you are moving faster than your friend at that moment, even though technically it could also be said that both of you have moved "0 kilometres over 0 hours."

The way we make this precise is suggested by your technique in the first scenario. To avoid dividing by zero, we let time elapse just a tiny bit, then measure how much the object's position has changed in that minute interval, and perform the division. Assuming we have arbitrarily sophisticated measuring instruments, we can keep taking smaller and smaller intervals, and arrive at something closer and closer to the "instantaneous velocity." But it is worth stating that "instantaneous velocity" is a theoretical concept, and not one that can be measured exactly. So when you are driving in a real car, and your speedometer reads some value, say, 127 kilometres per hour, what it's really doing is measuring your car's change in distance over a very small period of time, and reporting to you the quotient (not unlike what you were doing in that first scenario!).

Secants and tangents. The average rate of change of a function $f(x)$ between the input x and the input $x + h$ (for some small $h > 0$) is the total amount it changes, namely $f(x + h) - f(x)$, divided by the amount the input changed, namely $(x+h)-x = h$. If you graph the function $f(x)$ on the (x, y) plane, then geometrically what you are doing is drawing a line between the points $\big(x, f(x)\big)$ and $\big(x + h, f(x + h)\big)$ and calculating its *slope* (the "rise over run"). Such a line cutting through the function's graph is called a *secant line*, because it *cuts* through the function, at the very least at those two aforementioned points.

By taking h smaller and smaller (but never zero!) we continue to obtain secant lines, but the distance between the two points $\big(x, f(x)\big)$ and $\big(x+h, f(x+h)\big)$ will get smaller and smaller. You can imagine these points getting so close that it's as if the *line touches the function exactly once*, at the point $\big(x, f(x)\big)$ and nowhere else. This line, if it exists, will be called the *tangent line*, but we're not ready to precisely define it just yet.

All we note at this point is that if we admit the concept of the "instantaneous rate of change," and allow that it can be approximated by taking the the average rate of change over smaller and smaller intervals, then the slope of this tangent line (if it exists!) is exactly the instantaneous rate of change that we seek. So it seems that the slope of the tangent line should be the value of the quotient

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

when $h = 0$, if only we could plug in $h = 0$. But we cannot, since division by zero is undefined.

What we have learned from this preliminary section is that to understand rates of change, we shall have to develop a precise concept of what it means to take a function close to some input at which it is not necessarily defined. This is the notion of a *limit*.

2. Limits

02.IX In this section we establish the notion of a limit, which will be a rigorous way of formalising the idea of a function approaching a value at some point where it may or may not be defined. To understand the definition of a limit, it might be helpful to think about it as a generalisation of the concept of a continuous function f equalling some value y at the input $f(x)$.

Consider the very simple function $f(x) = x^2$. We know $f(3) = 3^2 = 9$. But since the function is continuous, it is also the case that if you want the output to be close to 9, then all you have to do is get the input close to 3. This is true in a *quantitative* sense!

If you want to get $f(x)$ to be within 0.1 of 9, then you just need to get x within 0.01 of the input value 3. How do we know this? Well $2.99^2 = 8.9401$ and $3.01^2 = 9.0601$, so (since the function x^2 is nondecreasing for positive inputs) for any x in the range $(2.99, 3.01)$, the value of $f(x)$ is in the range $(8.9, 9.1)$.

To sum up the previous paragraph in one sentence, you could say that in order for $f(x)$ to get 0.1-close to 9, it suffices to pick any input x that is 0.01-close to 3. But since $f(3) = 9$ and f is continuous, we can do much better. To guarantee that $f(x)$ be 0.01-close to 9, we just need to ensure that x is within 0.0001 of 3. And in fact, for any small number $\epsilon > 0$, there exists some δ such that whenever $x \in (3 - \delta, 3 + \delta)$, we must have $f(x) \in (9 - \epsilon, 9 + \epsilon)$. (In math, we often use the Greek letters ϵ and δ to denote very small numbers.)

The quintessential non-example. Here is an example to illustrate what we *don't* mean when we are talking about a function getting arbitrarily close to a number when its input gets close to some number. Let

$$f(x) = \sin\left(\frac{\pi}{x}\right);$$

this function is defined everywhere except at $x = 0$. The question is, does it "approach" anything when $x = 0$? This function oscillates between -1 and 1, but as x approaches 0 it oscillates very wildly. You might be fooled into thinking that as x approaches 0, the function f approaches 0. This is because

$$0 = \sin(\pi) = \sin(2\pi) = \sin(3\pi) = \cdots,$$

so $f(1) = 0$, $f(1/2) = 0$, $f(1/3) = 0$, and so on. But we also have

$$1 = \sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{9\pi}{2}\right) = \cdots,$$

giving us $f(2) = 1$, $f(2/5) = 1$, $f(2/9) = 1$, etc., and so this is a case of x approaching 0 but $f(x)$ *not* getting closer and closer to 0.

So the notion we end up defining for f getting close to something as x approaches something should not accidentally apply to this instance of f above. We are just about ready to state this notion formally.

The formal definition of a limit. First we define what a *neighbourhood* of a number is; it is simply some (open) interval containing that number. For example, the interval $(2.5, 3.5)$ is a neighbourhood of the number 3. So is $(2.9, 3.2)$, but $(10, 100)$ is not, and neither is $(1.5, 3)$, since we are considering open intervals.

Suppose that f is defined on all inputs in some neighbourhood of the number a, except possibly at a itself. (In other words, suppose that there is some neighbourhood of a contained in the domain of f, except possibly the point a.)

We say that the the *limit* of the function f at the input a is equal to L, and write

$$\lim_{x\to a} f(x) = L,$$

if for every $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $x \neq a$ and $x \in (a-\delta, a+\delta)$, we have $f(x) \in (L-\epsilon, L+\epsilon)$.

(If the use of the "element of" symbol $\in$ disturbs you, you can write the latter two expressions as $a-\delta < x < a+\delta$ and $L-\epsilon < f(x) < L+\epsilon$, respectively. Another way to write the definition of a limit is: For every $\epsilon > 0$ there is some $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have $\left|f(x) - L\right| < \epsilon$. Convince yourself that this means the exact same thing mathematically as the definition in terms of intervals that we had above—where is the condition $x \neq a$ encoded in the second definition?)

Because the formal definition of a limit is rather cumbersome, involving parameters ϵ and δ, in many cases we can make do with the following "simplified" or "intuitive" definition of a limit. *The limit of a function f at the input a is L if we can make $f(x)$ as close as we like to L by requiring that x be sufficiently close (but not necessarily equal to) a.*

In the example at the beginning of this section, we were convincing ourselves that

$$\lim_{x\to 3} x^2 = 9.$$

This should not be a mindblowing fact, since we of course have $3^2 = 9$. But we were careful in defining the concept of a limit in such a way that it now applies to many more situations as well, not just when $f(a)$ equals L.

A slightly harder example. Let us now show an example of a function f and an input a such that $\lim_{x\to a} f(x)$ exists and equals some number L, even when f is not defined at the input a itself. Suppose we let

$$f(x) = \frac{3x^2 - 2x - 1}{x - 1}.$$

Then we see that $f(x)$ is not defined at the input $x = 1$, since $1 - 1 = 0$ and dividing by 0 is illegal. Nevertheless, if you graph this function f using a graphing calculator or something like Desmos, you will see that the function f appears very simple. It is defined everywhere except at $x = 1$, and besides that little gap, its graph is simply a line; it is the unique line passing through the points $(0, 1)$ and $(2, 7)$. So even though f is not defined when $x = 1$, we might suspect from this picture that $\lim_{x\to 1} f(x) = 4$. Let us state this as a proposition and prove it rigorously (via the formal definition of a limit).

Proposition 1. *We have*

$$\lim_{x\to 1} \frac{3x^2 - 2x - 1}{x - 1} = 4.$$

Proof. Let $f(x) = (3x^2 - 2x - 1)/(x - 1)$ again for short. Suppose that $\epsilon > 0$ is given to us; we need to prescribe some $\delta > 0$ such that whenever $0 < |x-1| < \delta$, we have $0 < \bigl|f(x) - 4\bigr| < \epsilon$. We shall show that picking $\delta = \epsilon/3$ works.

Here's why. Assume that $0 < |x - 1| < \delta = \epsilon/3$. Multiplying the second inequality by 3 gives us

$$|3x - 3| < \epsilon.$$

We want a term of -4 inside the absolute-value bars, so let's just add and subtract 1 in that expression to make that happen:

$$|3x + 1 - 4| < \epsilon \tag{1}$$

Now we're almost done—we would be done if we had $f(x)$ instead of $3x+1$. But note that

$$3x^2 - 2x - 1 = (3x + 1)(x - 1),$$

so it's time to use the other assumption we had, which was $|x - 1| > 0$, to see that $(x - 1)/(x - 1) = 1$. In other words, under the assumption $|x - 1| > 0$, we have

$$3x + 1 = (3x + 1) \cdot 1 = \frac{(3x + 1)(x - 1)}{x - 1} = \frac{3x^2 - 2x - 1}{x - 1} = f(x).$$

Substituting this into (1) gives us $0 < \bigl|f(x) - 4\bigr| < \epsilon$, which is precisely what we wanted to show. ▮

Picking $\delta = \epsilon/3$ seemed to require a great deal of algebraic ingenuity here; let's try to convince ourselves that it's not so mysterious a choice. If you draw the graph of f, you'll note that the slope near $x = 1$ is 3 (well it's actually 3 everywhere!). So to guarantee no more than ϵ error in the output, we need to be three times as precise in the input: we can have no more than $\epsilon/3$ error in the input. This leads to the choice $\delta = \epsilon/3$. Note that the proof would also work if we picked δ even smaller, e.g., $\delta = \epsilon/10$. But $\delta = \epsilon/3$ is the largest one can take δ for the proof to work, and it is a common stylistic choice for a mathematician to pick a maximal value such as this.

Before moving on, we note here that you may at some point encounter authors who write "$f(x) \to L$ as $x \to a$." This is just an alternative way of writing $\lim_{x\to a} f(x) = L$. We will try to stick with the $\lim_{x\to a}$ version in these notes, as that is the convention the textbook uses, but there is a strong argument for the $f(x) \to L$ notation as well, as it makes it clear that $f(x)$ is not required to actually equal L at any stage, it just needs to get arbitrarily close to it.

04.IX **Formal negation of limit definition.** To fully understand the meaning of $\lim_{x\to a} f(x)$, it may be enlightening for us to consider the negation of this statement. Concretely, what does it mean for a number L *not* to be the limit of a function f at some input a? Negating long mathematical statements (such as the definition of limit) properly is not an easy task, and the untrained mathematician is liable to make errors especially concerning quantifiers such as "for all" and "there exists."

The correct formal negation is the following: *The real number L is not a limit of f at the input a if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there is some x in the domain of f satisfying $0 < |x - a| < \delta$ with the property that $\left|f(x) - L\right| \geq \epsilon$.*

This is a fairly convoluted statement, but if you ponder it for a while you will convince yourself that it does mean the opposite of what the limit definition says. When we write $\lim_{x\to a} f(x) = L$, we mean that you can get the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, and the opposite of this statement is there is some fixed positive distance ϵ such that even if you take x arbitrarily close to a, there will always be some badly-behaved choices of x that result in $f(x)$ being too far (at a distance at least ϵ) from $f(x)$.

(Note that the negation of the mathematical statement $\lim_{x\to a} f(x) = L$ should not be recorded as $\lim_{x\to a} f(x) \neq L$, since even writing down the left-hand side of this inequality suggests that it exists (is a real number), and as we shall see later in this section, the limit of a function f need not always exist at some given a.)

Here's an example to illustrate how you disprove the claim that the limit of a function f at some a is L.

Proposition 2. *Let*

$$f(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0 \end{cases}.$$

Then the limit of f at 0 is not 0.

Proof. We just need to find some $\epsilon > 0$ such that for any choice of $\delta > 0$, there will always be some real number x with $0 < |x - 0| < \delta$ but $\left|f(x) - 0\right| \geq \epsilon$. It turns out that choosing $\epsilon = 1/2$ works.

To see this, let $\delta > 0$ be given. We need to pick some x with $|x - 0| < \delta$ but with $\left|f(x) - 0\right| \geq 1/2$. This is not so hard—we have lots of choices! Choosing $x = \delta/2$ works, since $f(x) = f(\delta/2) = 1$, and $|1| \geq 1/2$.

Hence the limit of f at 0 is not 0. ∎

***A game with the Adversary.** Let f be a function, let L be a number, and let a be a number such that f is defined in some neighbourhood of a. The statement "$\lim_{x\to a} f(x) = L$" is either true or false (the "law of the excluded middle" tells us that every mathematical statement is either true or false). We can interpret its truthhood or falsity as the result of a game played between ourselves and a supernatural Adversary. Here's the how the game goes.

Game L (*The limit game*). Let f be a function, L a number, and a a number with a neighbourhood contained in the domain of f.

L1. [Adversary's turn.] The Adversary is allowed to pick any $\epsilon > 0$.

L2. [Our turn.] We can then pick any $\delta > 0$.

L3. [Adversary's turn.] The Adversary is then allowed to pick any x in the domain of f that satisfies

$$0 < |x - a| < \delta.$$

(This means that x is at distance less than δ from a, but doesn't equal a.)

L4. [Verdict?] If

$$\bigl|f(x) - L\bigr| < \epsilon,$$

that is, if the distance between $f(x)$ and L is less than ϵ, then we win; otherwise the Adversary wins. ▮

If $\lim_{x\to a} f(x) = L$ is true, then we can always win; the very definition of limit tells us that no matter what ϵ is chosen in step L1 by the Adversary, in step L2 there is always a choice of δ that will force the Adversary into letting us win in step L3. On the other hand, if $\lim_{x\to a} f(x) = L$ is false (either because this limit does not exist or because it equals something else), then the Adversary can always win. There is a choice of ϵ in step L1 such that no matter what we pick in step L2, the Adversary can beat us by making a further nefarious choice for x in step L3.

Nonexistent limits. We alluded earlier to the possibility of a function f not having any limits at some point a. We say that the limit of f at a *does not exist* if for all real numbers L, the limit of f at a is not L. In fact, the function f from the previous proposition has this property at $a = 0$. The proof is largely unchanged from the easier case above when we had just set L to 0, requiring only a little more care in the second paragraph.

Proposition 3. *For $x \neq 0$, let*

$$f(x) = \frac{|x|}{x}.$$

Then the limit of f as x approaches 0 does not exist.

Proof. We can re-express f by writing

$$f(x) = \begin{cases} -1, & \text{if } x < 0; \\ 1, & \text{if } x > 0 \end{cases},$$

which may be clearer to some readers (and also shows this is the same function from Proposition 2). Let L be arbitrary. We need to show that the limit of f at 0 is not L. To do this, we shall pick $\epsilon > 0$ such that for any choice of $\delta > 0$, there is some real number x with $0 < |x - 0| < \delta$ but $\bigl|f(x) - L\bigr| \geq \epsilon$. Just as in the previous proof for the special case $L = 0$, the choice $\epsilon = 1/2$ works.

Let $\delta > 0$ be given. We need to pick some x with $|x - 0| < \delta$ but with $\bigl|f(x) - L\bigr| \geq 1/2$. Here it matters whether L is positive or negative (if $L = 0$ then just appeal to the proof of Proposition 2). If L is positive, then choose $x = -\delta/2$. We have $0 < | - \delta/2 - 0| < \delta$ as required, and also

$$\bigl|f(-\delta/2) - L\bigr| = | - 1 - L| > | - 1 - 0| \geq 1/2 = \epsilon.$$

On the other hand, if L is negative, we instead choose $x = \delta/2$, so we still have $0 < |\delta/2 - 0| < \delta$ as needed, and this time

$$\bigl|f(\delta/2) - L\bigr| = |1 - L| > |1 - 0| \geq 1/2 = \epsilon.$$

Either way, we have satisfied the criterion of the limit at f at a not being L.

But the previous paragraph works for all numbers L, so the limit of f at a cannot equal any number! Hence we have proved that it does not exist. ∎

In the example just given, we now know that the limit of f at 0 does not exist. However, we often think of a limit intuitively as the function "approaching" some value, and in this sense it's wrong to say that f doesn't approach anything at all when the input gets close to 0. When you approach from the right, it is equal to 1. And when you approach from the left, the function equals -1. This leads us to the concept of a one-sided limit.

One-sided limits. Let f be a function, let L be a number, and let a be a number such that f is defined to the right of a. (More precisely, we assume that f is defined on the interval $(a, a+\eta)$ for some $\eta > 0$.) We say that *limit of $f(x)$ as x approaches a from the right is L*, and write

$$\lim_{x \to a^+} f(x) = L,$$

if for all $\epsilon > 0$ there is $\delta > 0$ such that any number x in the domain of f with

$$a < x < a + \delta$$

must satisfy

$$\left|f(x) - L\right| < \epsilon.$$

We might say in this case that L is the *right-hand limit* of f.

Likewise, let f be a function, L be a number, and suppose that a is such that f is defined to the *left* of a. We say that *limit of $f(x)$ as x approaches a from the left is L*, and write

$$\lim_{x \to a^-} f(x) = L,$$

if for all $\epsilon > 0$ there is $\delta > 0$ such that any number x in the domain of f with

$$a - \delta < x < a$$

must satisfy

$$\left|f(x) - L\right| < \epsilon.$$

If this is true then we say L is the *left-hand limit* of f.

More loosely, L is the right-hand limit of f at a if you can get $f(x)$ to be arbitrarily close to L by requiring that x be sufficiently close to *a and greater than a*, and it's the left-hand limit of f at a if you can get $f(x)$ as close as you like to L by requiring that x be sufficiently close to *a and smaller than a*.

Both of these definitions are relaxations of the definition of limit; it is "easier" to be a left-hand limit than it is to be a *bona fide* limit). For example, we saw in Proposition 3 that 1 is not a limit of f from that proposition, because

f has no limits. However, once you have digested the definitions of one-sided limits, you should be able to check that 1 is the right-hand limit of f at 0, and -1 is the left-hand limit at 0.

If the actual limit of f at a point a is L, then it is a consequence of the definitions that the right-hand limit and the left hand limit are also L. Conversely, if the one-sided limits both equal the same L, then the limit is actually L. This can be summed up symbolically in the following statement, which we'll call a theorem since it comes in handy now and again.

Theorem 4. *For any function f and numbers a and L, we have*

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^+} f(x) = L \qquad \textit{and} \qquad \lim_{x\to a^-} f(x) = L. \quad \blacksquare$$

A simple application of Theorem 4 is to prove that the function $f(x) = |x|$ goes to 0 as x approaches 0. We'll see this as an example in the next section, but it's really not a difficult proof, so you could try it now. Show that both one-sided limits approach 0, then simply invoke the theorem.

Here's a related theorem (no less useful) that tells us what happens if the one-sided limits of a function do not agree at some given point. It shows that the example exhibited by Proposition 3 is an instance of a very general phenomenon.

Theorem 5. *If*

$$\lim_{x\to a^+} f(x) = L, \qquad \lim_{x\to a^-} f(x) = M,$$

and $L \neq M$, then $\lim_{x\to a} f(x)$ does not exist.

Proof. We skip this proof as it's beyond the scope of this course (you would see it in, e.g., MATH 242 or MATH 254), but it is not so different in spirit to the proof of Proposition 3. The motivated student is encouraged to try it as an advanced exercise. (If you try for a while and get stuck, email me or come to office hours with what you have and I could help you fill in the details.) $\blacksquare$

If we were told Theorem 5 from the start, we could have used it to prove Proposition 3 without much ado.

Proof of Proposition 3 using Theorem 5. Recall that we are trying to prove that the limit of the function $f(x) = |x|/x$ does not exist as x approaches 0. Since $|x| = x$ for all $x > 0$, we see that

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1.$$

(The last equality is easy to see if you just graph the constant function 1. If you are a serious student and insist on a rigorous proof, then please do Exercise 24 of

Section 2.4 in the textbook, which invites you to furnish a proof that $\lim_{x\to a} c = c$ for all numbers a and c.)

On the other hand, since $|x| = -x$ for all $x < 0$, we have

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^+} -1 = -1.$$

So the left- and right-hand limits do not agree, and Theorem 5 applies in this situation to tell us that the limit does not exist. ▮

One-sided limits not agreeing is one way that the limit of a function at a given point can fail to exist. Another way is for the limit to get arbitrarily large, which is the situation we tackle next.

09.IX **Infinite limits and vertical asymptotes.** We say that the *limit of $f(x)$ is infinity as x approaches a from the right*, and write

$$\lim_{x\to a^+} f(x) = \infty,$$

if for every large (positive) number M, there exists $\delta > 0$ such that every x satisfying

$$a < x < a + \delta$$

has

$$f(x) > M.$$

If, instead, for every large positive number M there is $\delta > 0$ such that every x with

$$a < x < a + \delta$$

satisfies

$$f(x) < -M,$$

then we say that the limit of $f(x)$ is *minus infinity* as x approaches a from the right, and write

$$\lim_{x\to a-} f(x) = -\infty.$$

The same definition can be made for left-hand limits by replacing the assumption $a < x < a + \delta$ with the assumption $a - \delta < x < a$ above. This makes for four situations in total, and if any one of these situations occurs, then f is said to have a *vertical asymptote* at the line $x = a$.

Just as in the case of finite limits, when the left- and right-hand limits agree (and are both the same type of infinity), then we simply say that the *limit* is infinity (or minus infinity, accordingly). In other words, we have

$$\lim_{x\to a} f(x) = \infty$$

if and only if

$$\lim_{x\to a^-} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = \infty,$$

and we have

$$\lim_{x\to a} f(x) = -\infty$$

if and only if

$$\lim_{x\to a^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = -\infty.$$

(Let p and q be mathematical statements. The phrase "p if q" means that q implies p, and the phrase "p only if q" means that p implies q—think about it for a second, it makes sense! So the statement "p if and only if q" means that p is true exactly when q is true, and vice versa.)

One should not interpret the symbol ∞ above as a number, since there is no such number. When we write $\lim_{x\to a} f(x) = \infty$, we simply mean that this expression can be made arbitrarily large as by requiring that x be sufficiently close to a. It should also be stressed that the equality symbol in this expression should *not* be interpreted as meaning that the limit of f at a exists, because the terminology of a limit "existing" is reserved for when the limit equals a number (and, once again, ∞ is not a number). As with finite limits, sometimes authors write, e.g., $f(x) \to -\infty$ as $x \to a$ instead of $\lim_{x\to a} f(x) = -\infty$. This notation is also valid, and in some sense preferable as it does not suggest that $f(x)$ ever "equals" some "number" ∞.

Let's look at a couple of simple examples of functions with vertical asymptotes.

Proposition 6. *We have*

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

and

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Proof. We prove both statements separately. The first asserts that $1/x$ can be made arbitrarily large by taking $x \to 0$ from the right. To prove it, let $M > 0$ be an arbitrarily large number (given to us). Pick $\delta = 1/M$, and assume that $0 < x < \delta = 1/M$. Because $x > 0$, we are allowed to write $1/x$, and the fact that $x < 1/M$ implies that $1/x > M$, which is what we needed to prove in the definition of infinite limit.

The second statement is proved extremely similarly, but we include the full proof for completeness. Let M be given, and pick $\delta = 1/M$. Assume that $-\delta < x < 0$; the right inequality here assures us that the expression $1/x$ is well-defined, and the left inequality here can be rewritten as $x > -\delta = -1/M$. Rearranging the equation now yields $1/x < -M$, which is what we needed to show in order to prove that the limit is $-\infty$. ∎

To sum up this proof, we can make $1/x$ arbitrarily large to the right of 0 by requiring that $x > 0$ be sufficiently close to 0, and we can make $1/x$ an arbitrarily large negative number by requiring that $x < 0$ be sufficiently close to (the left of) 0.

Here's another example where the left- and right-hand limits agree on the type of infinity.

Proposition 7. *The limit of*

$$f(x) = \frac{1}{(x-3)^2}$$

is infinity as x approaches 3.

Proof. We can do the left- and right-hand limits separately, but it is not hard to prove this whole thing all at once (i.e., to consider both sides of 3 at the same time).

Let $M > 0$ be given. We want to prescribe $\delta > 0$ such that all $x \neq 3$ with

$$3 - \delta < x < 3 + \delta$$

satisfy

$$\frac{1}{(x-3)^2} > M.$$

(The assumption $3 - \delta < x < 3 + \delta$ is what we mean by considering both sides of 3 at the same time; note that this assumption together with the assumption that $x \neq 3$ can be summed up in the single inequality $0 < |x - 3| < \delta$.) Setting $\delta = 1/\sqrt{M}$ works this time, so our assumption on x will be

$$3 - \frac{1}{\sqrt{M}} < x < 3 + \frac{1}{\sqrt{M}},$$

and that $x \neq 3$ of course, so that $f(x)$ is even defined. The difference $x - 3$ may be positive or negative; we don't know which. But what we do know, from our assumption, is that the distance $|x - 3|$ between x and 3 satisfies

$$|x - 3| < \frac{1}{\sqrt{M}}.$$

Squaring both sides (and using the fact that $|x-3|^2 = (x-3)^2$, since the square of a negative number is positive anyway) gives us

$$(x-3)^2 < \frac{1}{M},$$

and rearranging yields

$$f(x) = \frac{1}{(x-3)^2} > M,$$

which is what we needed to show. (The number M was arbitrary in this proof, so we can keep repeating with larger and larger M to show that $f(x)$ can be made arbitrarily large in the way shown above.) ∎

Once again, picking $\delta = 1/\sqrt{M}$ made the algebra work out nicely, but you could have written a valid proof by picking, say, $\delta = 1/M$ instead, but then there'd be some issues with having to assume that $M > 1$. (Morally this is O.K., since we're only worried about what happens when M is very large, but *technically* our definition had $M > 0$.) Don't stress too much about picking the "best" value for δ. Generally, you find δ by first trying out various values of M and seeing what δ is necessary to deal with that M; after a few tries you should be able to come up with a formula that works generally. The broader thing to take away from reading this proof or trying to do it yourself is that no matter what M is given, you can pick *some* δ that makes the proof work out.

11.IX **Limits at infinity and horizontal asymptotes.** There is one last type of limit that remains to be defined. This is the case in which the values of $f(x)$ can be taken arbitrarily close to a value by making x sufficiently large, either in the positive direction or the negative direction. Formally, we write

$$\lim_{x\to\infty} f(x) = L$$

and say that the *limit of $f(x)$ is L as x approaches infinity* if for all $\epsilon > 0$ there exists some large $M > 0$ such that every x with $x > M$ has $\bigl|f(x) - L\bigr| < \epsilon$. Likewise, we say the *limit of $f(x)$ is L as x approaches negative infinity* and write

$$\lim_{x\to-\infty} f(x) = L$$

if the same condition applies, except the inequality $x > M$ is replaced by $x < -M$. In either of these situations, we say that $y = f(x)$ has a *horizontal asymptote* at the line $y = L$.

Note that the same example from Proposition 6 also exhibits a horizontal asymptote.

Proposition 8. *We have*

$$\lim_{x\to\infty} 1/x = 0 \qquad \textit{and} \qquad \lim_{x\to-\infty} 1/x = 0.$$

Proof. Let $\epsilon > 0$ and pick $M = 1/\epsilon$. Then any number x with $x > M$ has

$$\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{M} = \epsilon,$$

so $\lim_{x\to\infty} 1/x = 0$.

For the other inequality, the proof is very similar. Let $\epsilon > 0$ and again we choose $M = 1/\epsilon$. Now any (negative) number x with $x < -M$ has

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{-x} < \frac{1}{M} = \epsilon,$$

so $\lim_{x\to-\infty} 1/x = 0$. ▮

Since ∞ is not a number, it does not make sense to write an expression such as, say, $1/\infty$. On the other hand, you may have heard people say, informally, something like "anything divided by infinity is zero." This statement can be made into a proper mathematical one by expressing it in terms of limits. It is true that for any fixed number K, the limit $\lim_{x\to\infty} K/x$ equals 0. (As an exercise, prove this directly by very slightly modifying the proof of Proposition 8 above. Hint: The proposition already gives a proof when $K = 1$. So for the general case, instead of setting $M = 1/\epsilon$, you might try choosing something very similar but involving K.)

Infinite limits at infinity. As with limits as x approaches some finite a, the infinite limit is said not to exist if it does not equal any finite number L. There is one special case of this worth mentioning, which is when the function $f(x)$ can be made arbitrarily large (positive or negative) by requiring that x be sufficiently large (positive or negative). Hence there are four cases of infinite limits at infinity, namely

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = \infty,$$

$$\lim_{x\to\infty} f(x) = -\infty, \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -\infty.$$

Tedious as it is, let us formally define all four of these statements, for the sake of completeness.

a) We say that $f(x)$ *goes to infinity as x goes to infinity* and write

$$\lim_{x\to\infty} f(x) = \infty$$

if for all $M > 0$, there exists $N > 0$ such that for all x with $x > N$, we have $f(x) > M$.

b) We say that $f(x)$ *goes to infinity as x goes to negative infinity* and write

$$\lim_{x\to-\infty} f(x) = \infty$$

if for all $M > 0$, there exists $N > 0$ such that for all x with $x < -N$, we have $f(x) > M$.

c) We say that $f(x)$ *goes to negative infinity as x goes to infinity* and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if for all $M > 0$, there exists $N > 0$ such that for all x with $x > N$, we have $f(x) < -M$.

d) We say that $f(x)$ *goes to negative infinity as x goes to negative infinity* and write

$$\lim_{x\to-\infty} f(x) = -\infty$$

if for all $M > 0$, there exists $N > 0$ such that for all x with $x < -N$, we have $f(x) < -M$.

That was a mouthful. But by this point it is hoped that the reader would be comfortable enough with these formal definitions to be able to formulate, say (a), (c), and (d) after being shown (b).

We finish off this very long section by presenting an example of a function that seems to grow fairly slowly $x \to \infty$, but nevertheless still goes to infinity as x is taken to be large.

Proposition 9. *We have*

$$\lim_{x\to\infty} \sqrt[3]{x} = \infty$$

and

$$\lim_{x\to-\infty} \sqrt[3]{x} = -\infty.$$

Proof. Let $M > 0$ be a large positive number given to us. We want to pick N such that for all $x > N$, we have $\sqrt[3]{x} = x^{1/3} > M$. After a moment's pause we see that choosing $N = M^3$ works, since if we assume that $x > N$, then

$$x^{1/3} > N^{1/3} = (M^3)^{1/3} = M.$$

The negative infinite limit is proved similarly; we choose $N = M^3$ this time as well, so that whenever $x < -N$, we have

$$x^{1/3} < (-N)^{1/3} = (-M^3)^{1/3} = -M.$$

This completes the proof. ▮

3. Rules for evaluating limits

In the previous section, we already encountered two rules (namely, Theorems 4 and 5) that can help us find limits of functions or determine that they do not exist. This section will be devoted to establishing a few more useful laws. First we record a list of algebraic laws, which show that the limit behaves well under a number of algebraic operations such as addition, multiplication, etc.

Theorem 10 (*Algebraic limit laws*). *Let c be any fixed number, and suppose that f and g are functions such that*

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

both exist. Then

i) $\lim_{x\to a}\big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;

ii) $\lim_{x\to a}\big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;

iii) $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x)$;

iv) $\lim_{x\to a}\big(f(x)g(x)\big) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$;

v) $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f(x)\big/\lim_{x\to a} g(x)$, *provided that* $\lim_{x\to a} g(x) \neq 0$;

vi) for all positive integers n, $\lim_{x\to a} f(x)^n = \big(\lim_{x\to a} f(x)\big)^n$; *and*

vii) for all positive integers n, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *(provided that* $\lim_{x\to a} f(x) > 0$ *in the case that* n *is even ,so that the* n*th root is well-defined).*

These statements are all true even when $x \to a$ *is replaced with* $x \to \infty$, $x \to -\infty$, $x \to a^+$, *or* $x \to a^-$.

The identities above are known as (i) the sum law, (ii) the difference law, (iii) the constant multiple law, (iv) the product law, (v) the quotient law, (vi) the power law, and (vii) the root law, respectively. They can be expressed in words as well: the limit of a sum is the sum of the limits, the limit of a difference is the difference of the limits, and so on.

14.IX *Proof.* We only prove (i) in these notes. Laws (ii) through (v) are proved in Appendix F of the textbook. Law (vi) is a consequence of repeating law (iv) n times (formally, you would need to use the *principle of mathematical induction*, which is outside the scope of this course), and the proof of law (vii) is assigned to you in the textbook as Exercise 69 of Section 2.5. All of these proofs only deal with the case that $x \to a$ for some finite number a, as opposed to $x \to \infty$; it is an instructive exercise to try and adapt the proof below for the $x \to \infty$ case, to test that you have fully digested this proof and the definition of limits at infinity.

Enough talk. On to the proof of (i). We shall make use of the *triangle inequality*, which says that for any numbers a, b, and c, we have

$$|a-c| \leq |a-b| + |b-c|.$$

In other words, the length of any side in a triangle is always smaller than the sum of the other two sides.

Suppose $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. We want to show that the limit of $f(x)+g(x)$ as $x \to a$ is $L_1 + L_2$. Let $\epsilon > 0$. We wish to pick a $\delta > 0$ such that for all x with $0 < |x-a| < \delta$, we have

$$\big|f(x) + g(x) - (L_1 + L_2)\big| < \epsilon.$$

Well, since $\lim_{x\to a} f(x) = L_1$, we can make $f(x)$ arbitrarily close to L_1 by prescribing that x be sufficiently close to a. So there is some $\delta_1 > 0$ such that if we assume $0 < |x-a| < \delta_1$, then

$$\big|f(x) - L_1\big| < \frac{\epsilon}{2}.$$

(We choose $\epsilon/2$ so that there is some additional room for error when we deal with g.) Likewise, there is some $\delta_2 > 0$ such that if we assume $0 < |x - a| < \delta_2$, then

$$\big|g(x) - L_2\big| < \frac{\epsilon}{2}.$$

Now, set $\delta = \min\{\delta_1, \delta_2\}$. This guarantees both $\delta \le \delta_1$ and $\delta \le \delta_2$, so that if we assume $0 < |x - a| < \delta$, then the inequalities

$$0 < |x - a| < \delta_1 \qquad \text{and} \qquad 0 < |x - a| < \delta_2$$

both hold. As a result, by our observations in the previous paragraph, both of the inequalities

$$\big|f(x) - L_1\big| < \frac{\epsilon}{2} \qquad \text{and} \qquad \big|g(x) - L_2\big| < \frac{\epsilon}{2}$$

are true as well. Finally we see that

$$\begin{aligned}\big|f(x) + g(x) - (L_1 + L_2)\big| &= \big|f(x) - L_1 + g(x) - L_2\big| \\ &\le \big|f(x) - L_1\big| + \big|g(x) - L_2\big| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

where the second line follows from the aforementioned triangle inequality. This completes the proof. ▮

These laws are extremely powerful in general, allowing us to express the limit of a complex expression in terms of the limits of simpler expressions. For example, using laws (i), (iii), and (iv), we can deduce that

$$\lim_{x \to 3} 4x^2 - 7 = 4 \cdot \Big(\lim_{x \to 3} x\Big)\Big(\lim_{x \to 3} x\Big) - \lim_{x \to 3} 7. \tag{2}$$

We already encountered an expression of the form $\lim_{x \to a} c$, where c is a constant number, in an earlier proof. It should be obvious that this limit is c, and that $\lim_{x \to a} x = a$. Let us state these two facts, along with two more as a theorem, so that we can easily invoke them later on. (We continue the numbering from Theorem 10 in order to remain consistent with the textbook.)

Theorem 11 (*Special limits*). *For any positive integer n and any numbers a and c,*

viii) $\lim_{x \to a} c = c$;

ix) $\lim_{x \to a} x = a$;

x) $\lim_{x \to a} x^n = a^n$; *and*

xi) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$.

Laws (viii) and (ix) remain true when $x \to a$ is replaced by $x \to \infty$ or $x \to -\infty$.

Proof. As mentioned before, the proof of law (viii) is presented to you as Exercise 24 in Section 2.4; the proof of law (ix) is Exercise 23 in the same section. This is in the case where x approaches some finite a; you might like to prove the corresponding limits at infinity to test your understanding of those definitions.

Laws (x) and (xi) are easy to prove once we have all the other laws established. To prove law (x), apply the power law with $f(x) = x$ and then law (ix) to obtain

$$\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n.$$

To prove law (xi), apply the root law with $f(x) = x$ and then law (ix) to obtain

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x} = \sqrt[n]{a}. \quad \blacksquare$$

With these new laws in hand, we can revisit the polynomial $4x^2 - 7$ from earlier and conclude that

$$\lim_{x\to 3}(4x^2 - 7) = 4 \cdot 3^2 - 7 = 29.$$

In fact, we don't even need to break up x^2 into $x \cdot x$ like we did in (2), as law (x) takes care of monomials such as these.

When applying the limit laws to a determine what happens when $x \to 3$ to a simple polynomial such as $4x^2 - 7$, it seemed like we could just substitute in $x = 3$ in the expression and get the right answer. This turns out to be true in the general case $x \to a$ as long as a is in the domain of f.

Theorem 12 (*Direct substitution*). *Let f be a polynomial or a rational function (a quotient of two polynomials). If a is in the domain of f, then*

$$\lim_{x\to a} f(x) = f(a).$$

Proof. Every rational function f is of the form

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0},$$

where m and n are nonnegative integers, and $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n$ are real (possibly negative or zero) coefficients. Taking the limit of both sides, the quotient law deals with the fraction, the sum law splits the limit of the sum into the sum of limits, and from there we use law (x) and the constant multiple law to break up these terms further. We then apply laws (viii) and (ix) as to finish off the proof. $\blacksquare$

This theorem applies to functions involving roots as well, thanks to the root law. (It more generally applies to all *continuous* functions—we will discuss the notion of continuity very soon.) Using the direct substitution rule is fairly easy whenever its hypotheses are satisfied, so we will not dwell on this. There is, however, a more advanced version of this rule that is useful in certain cases, such as when $f(x)$ is a rational function and a is *not* in the domain of f (e.g., because plugging in $x = a$ makes the denominator 0). Then we may have to invoke the following theorem in order to use the direct substitution rule.

Theorem 13. *Let a be a number, and suppose that f and g are two functions that are equal for all x in some interval containing a, except possibly at a itself. Then if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, we must have*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

**Proof.* This proof is a bit more advanced than you should fairly be expected to follow in this course, though if you take a second-year mathematics class such as MATH 242 one day, you'd find it a routine exercise by the end of that course. We include it in these notes for completeness.

We know that f and g agree in some interval containing a, except possibly at a itself. To be concrete, this means there exists some number $\eta > 0$ be such that $f(x) = g(x)$ for every x with $0 < |x-a| < \eta$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} = M$. We want to show that $L = M$; we shall show that $|L - M| < \epsilon$ for all $\epsilon > 0$, which is an equivalent statement (a nonnegative quantity that is less than any positive quantity must be zero).

Let $\epsilon > 0$. We will use an "$\epsilon/2$ trick" again, just as in the proof of the sum law above. Since $\lim_{x\to a} f(x) = L$, there is δ_1 such that for all x with $0 < |x - a| < \delta_1$, we have

$$\bigl|f(x) - L\bigr| < \epsilon/2. \tag{3}$$

Likewise, since $\lim_{x\to a} g(x) = M$, there is δ_2 such that for all x with $0 < |x-a| < \delta_2$, we have

$$\bigl|g(x) - M\bigr| < \epsilon/2. \tag{4}$$

Now pick $\delta = \min\{\eta, \delta_1, \delta_2\}$, so that every x with $0 < |x-a| < \delta$ has $f(x) = g(x)$ *and* satisfies both (3) and (4). We now combine all three of these facts and invoke the triangle inequality to obtain

$$\begin{aligned} |L - M| &= \bigl|L - g(x) + g(x) - M\bigr| \\ &\le \bigl|L - g(x)\bigr| + \bigl|g(x) - M\bigr| \\ &= \bigl|f(x) - L\bigr| + \bigl|g(x) - M\bigr| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

As we said above, since this proof shows that $|L - M| < \epsilon$ for every $\epsilon > 0$, we must have $|L - M| = 0$; that is, $L = M$. ▮

If we'd had these theorems from the start, we wouldn't have had to use so much elbow grease in proving Proposition 1. Here's how we could have proceeded.

Proof of Proposition 1 using Theorems 12 and 13. Let

$$f(x) = \frac{3x^2 - 2x - 1}{x - 1};$$

recall that we are trying to show that $\lim_{x\to 1} f(x) = 4$. We factor $3x^2 - 2x - 1 = (3x+1)(x-1)$ and observe that for any $x \neq 1$,

$$f(x) = \frac{3x^2 - 2x - 1}{x - 1} = 3x + 1.$$

Letting $g(x) = 3x+1$, on any open interval containing 1 we have $f(x) = g(x)$ everywhere except at $x = 1$, so Theorem 13 applies to inform us that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$. Moreover, $g(x)$ is a polynomial with $x = 1$ in its domain, so

$$\lim_{x\to 1} g(x) = 3 \cdot 1 + 1 = 4$$

by Theorem 12. ∎

Here's another example in which we use the root law.

Proposition 14. *We have*

$$\lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} = \frac{1}{4}.$$

Proof. We cannot use the quotient law, since $x^2 \to 0$ as $x \to 0$. If we can massage the expression to obtain a factor of x^2 in the numerator, then (in an open interval containing 0 but not at 0 itself), we can cancel the x^2 in the denominator and perhaps make some headway. This strategy is facilitated by the observation that

$$\left(\sqrt{x^2+4} - 2\right)\left(\sqrt{x^2-4} + 2\right) = \left(\sqrt{x^2-4}\right)^2 - 2^2 = x^2.$$

Hence we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} \cdot \frac{\sqrt{x^2+4} + 2}{\sqrt{x^2+4} + 2} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4} + 2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+4} + 2}.
\end{aligned}$$

We no longer have the issue of the limit of the denominator being 0, and can happily apply the algebraic limit laws as well as the direct substitution property:

$$\begin{aligned}\lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2} &= \frac{1}{\lim_{x\to 0}\bigl(\sqrt{x^2+4}+2\bigr)}\\ &= \frac{1}{\lim_{x\to 0}\sqrt{x^2+4}+\lim_{x\to 0}2}\\ &= \frac{1}{\sqrt{\lim_{x\to 0}(x^2+4)}+2}\\ &= \frac{1}{\sqrt{4}+2}\\ &= \frac{1}{4} \quad ▮\end{aligned}$$

Using one-sided limits. Recall Theorem 4, which states that a limit exists and equals some L if and only if both one-sided limits exist and equal L. This can be used to compute $\lim_{x\to 0}|x|$, even though the algebraic limit laws do not apply to the function $|x|$ directly.

Proposition 15. *We have*

$$\lim_{x\to 0}|x| = 0.$$

Proof. Towards the invocation of Theorem 4, we calculate the one-sided limits of $|x|$ using the direct substitution rule. When $x > 0$, we have $|x| = x$, so

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0,$$

since For all $x < 0$ we instead have $|x| = -x$, so

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-} -x = 0.$$

Both one-sided limits agree, so we conclude that the limit is 0. ▮

Evaluating limits at infinity. Recall that most of the algebraic limit laws established earlier in this section apply to limits at infinity. Hence we can combine the root and power laws with Proposition 8 to prove the following useful theorem.

Theorem 16. *Let $r > 0$ be a rational number. Then*

$$\lim_{x\to\infty}\frac{1}{x^r} = 0.$$

If $r > 0$ is such that x^r is defined for negative values of x as well, then

$$\lim_{x\to-\infty}\frac{1}{x^r} = 0.$$

Proof. Let $r > 0$ be a rational number. This means there are positive integers m and n such that $r = m/n$, so we can write

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to\infty} \sqrt[n]{\left(\frac{1}{x}\right)^m} = \sqrt[n]{\left(\lim_{x\to\infty} \frac{1}{x}\right)^m} = 0,$$

where the third equality follows from the root and power laws, and the last follows from Proposition 8.

This works for $x \to -\infty$ as well, as long as x^r is defined for negative values of x. (If n is even and m is odd, x^m is negative and the nth root of a negative is not defined, so it does not makes sense to write $x^r = \sqrt[n]{x^m}$ in this case.) ∎

When studying the behaviour of a function $f(x)$ as $x \to \infty$, we can always multiply the whole expression by x^r/x^r for any r, since we are interested in the case where x is large and don't have to worry about the denominator possibly being 0. This is very useful in practice when applying Theorem 16.

Here's a simple example.

16.IX **Proposition 17.** *We have*

$$\lim_{x\to\infty} \frac{5x^2 - 3x + 4}{3x^2 + 10x - 3} = \frac{5}{3}$$

Proof. Since we're only interested in the behaviour when x is very large, we can assume $x > 0$, so that x^2/x^2 is defined. We divide by this function, obtaining

$$\lim_{x\to\infty} \frac{5x^2 - 3x + 4}{3x^2 + 10x - 3} = \lim_{x\to\infty} \frac{(5x^2 - 3x + 4)/x^2}{(3x^2 + 10x - 3)/x^2} = \lim_{x\to\infty} \frac{5 - 3/x + 4/x^2}{3 + 10/x - 3/x^2}.$$

Invoking the quotient and sum laws now, we take the limits of each term in the numerator and denominator individually, and Theorem 16 applies to give us

$$\lim_{x\to\infty} \frac{5 - 3/x + 4/x^2}{3 + 10/x - 3/x^2} = \lim_{x\to\infty} \frac{5 - 0 + 0}{3 + 0 - 0} = \frac{5}{3}.$$

(We technically used the constant multiple law here too, e.g., when performing the operation $\lim_{x\to\infty} 4/x^2 = 4 \cdot \lim_{x\to\infty} 1/x^2$.) ∎

The limit as $x \to -\infty$ of the function in the above proposition is also $5/3$ (work it out!), but it is not always the case that the limits as $x \to \infty$ and $x \to -\infty$ agree.

Proposition 18. *Let*

$$f(x) = \frac{\sqrt{2x^2 + 1}}{5x - 1}.$$

We have

i) $\lim_{x\to\infty} f(x) = \sqrt{2}/5$;

ii) $\lim_{x\to-\infty} f(x) = -\sqrt{2}/5$*;*

iii) $\lim_{x\to(1/5)^+} f(x) = \infty$*; and*

iv) $\lim_{x\to(1/5)^-} f(x) = -\infty$*.*

Proof. To prove part (i), we perform the same strategy. The largest power of x in the denominator is just x, so we divide top and bottom by x, noting that for $x > 0$ one has $x = \sqrt{x^2}$:

$$\begin{aligned}\lim_{x\to\infty} \frac{\big(\sqrt{2x^2+1}\big)/x}{(5x-1)/x} &= \lim_{x\to\infty} \frac{\big(\sqrt{2x^2+1}\big)/\sqrt{x^2}}{5-1/x}\\ &= \lim_{x\to\infty} \frac{\big(\sqrt{2+1/x^2}\big)}{5-1/x}\\ &= \frac{\sqrt{2+0}}{5-0}\\ &= \frac{\sqrt{2}}{5}\end{aligned}$$

It looks like we can do the same thing when $x < 0$, and morally we can, except we have to remember that for negative x one instead has $x = -\sqrt{x^2}$. So

$$\begin{aligned}\lim_{x\to-\infty} \frac{\big(\sqrt{2x^2+1}\big)/x}{(5x-1)/x} &= \lim_{x\to-\infty} \frac{\big(\sqrt{2x^2+1}\big)/\big(-\sqrt{x^2}\big)}{5-1/x}\\ &= \lim_{x\to-\infty} \frac{-\big(\sqrt{2+1/x^2}\big)}{5-1/x}\\ &= \frac{-\sqrt{2+0}}{5-0}\\ &= \frac{-\sqrt{2}}{5},\end{aligned}$$

settling claim (ii). We might as well take a moment here to practise finding infinite limits. Note that $x = 1/5$ is outside the domain of f, since it makes the denominator 0. Note that when $x = 1/5$, the numerator is a positive number (since squaring a positive number preserves its positivity, and all the other operations in that expression do as well—if you were extra insecure, you'd calculate the numerator fully and see that it equals $\sqrt{27}/5$, a positive quantity). Hence slightly to the left of $x = 1/5$ and slightly to the right, f is also positive. When x just slightly to the right of $1/5$, the denominator is a very small positive number, and we can make this denominator arbitrarily small by taking x sufficiently close to x from the right. So

$$\lim_{x\to(1/5)^+} = \infty,$$

since the numerator is positive as well. On the other hand, when x is very close to $1/5$ from the left, the denominator is a very small negative number, and again this can be taken arbitrarily close to 0 by taking x sufficiently close to x from the left. So

$$\lim_{x\to(1/5)^-} = -\infty. \qquad \blacksquare$$

The examples we've seen of limits at infinity have been finite, so before moving on, let's see an example where the limit at infinity is infinite.

Proposition 19. *We have*

$$\lim_{x\to\infty} x - x^2 = -\infty.$$

Proof. We might be tempted to try the difference law here, but since the limits are both infinite we get $\infty - \infty$, which is nonsense. A better thing to do is to factor $x - x^2 = x(1-x)$. When x gets large, x is a large positive number and $(1-x)$ is a large negative number. Hence the product is a (very) large negative number. We conclude that the limit must be $-\infty$. $\blacksquare$

The squeeze theorem. We end this section by highlighting the following intuitive theorem, which also goes by the names *sandwich theorem* or *two policemen and a drunk theorem.*

Theorem 20 (*Squeeze theorem*). *Let a be a number and suppose that for all $x \neq a$ in an open interval containing a, we have*

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x)$ exists and equals L.

Proof. This proof is given in Appendix F of the textbook, but we include it in the notes as well for completeness.

Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is $\delta_1 > 0$ such that every x with $0 < |x-a| < \delta_1$ satisfies

$$\big|f(x) - L\big| < \epsilon. \tag{5}$$

Likewise, there is $\delta_2 > 0$ such that every x with $0 < |x-a| < \delta_2$ satisfies

$$\big|h(x) - L\big| < \epsilon. \tag{6}$$

Let $\eta > 0$ be a number such that for all x with $0 < |x-a| < \eta$, we have $f(x) \leq g(x) \leq h(x)$. Subtracting L from this inequality gives us

$$f(x) - L \leq g(x) - L \leq h(x) - L \tag{7}$$

for all such x. Hence, letting $\delta = \min\{\delta_1, \delta_2, \eta\}$, every x with $0 < |x-a| < \delta$ satisfies all three of (5), (6), and (7). Combining these three inequalities yields

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon,$$

which implies that

$$\bigl|g(x) - L\bigr| < \epsilon,$$

as desired. ▮

Near the beginning of the course we encountered the pathological function $\sin(\pi/x)$ and convinced ourselves that it does not possess a limit as $x \to 0$. However, by combining the squeeze theorem with Proposition 15, we can show that a modification of $\sin(\pi/x)$ does in fact approach a limit.

Proposition 21. *We have*

$$\lim_{x\to 0} x\sin\Bigl(\frac{\pi}{x}\Bigr) = 0.$$

Proof. It looks like this follows from the product law, but we cannot actually apply that law because $\lim_{x\to 0}\sin(\pi/x)$ does not exist.

The thing to note is that the function sin is bounded from above by the constant function 1 and bounded from below by the constant function -1. In other words, for all real numbers t we have

$$-1 \le \sin(t) \le 1.$$

So for all $x > 0$ we have

$$-x \le x\sin\Bigl(\frac{\pi}{x}\Bigr) \le x$$

and for all $x < 0$ we have

$$x \le x\sin\Bigl(\frac{\pi}{x}\Bigr) \le -x.$$

In other words, the inequality

$$-|x| \le x\sin\Bigl(\frac{\pi}{x}\Bigr) \le |x|$$

holds for all $x \ne 0$. By Proposition 15, $\lim_{x\to 0}|x| = 0$, so we also have $\lim_{x\to 0} -|x| = 0$ by the constant multiple law. Hence by the squeeze theorem (with $f(x) = -|x|$, $g(x) = x\sin(\pi/x)$, and $h(x) = |x|$), we have

$$\lim_{x\to 0} x\sin\Bigl(\frac{\pi}{x}\Bigr) = 0.$$

which is what we wanted to show. ▮

We will see more applications of the squeeze theorem in the future, for example, when we work out the derivative (whatever that is) of the sine function.