A Fibonacci Proof

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Theorem. *For all $n \in \mathcal{N}$, the n-th number in the Fibonacci sequence* $(1, 1, 2, 3, 5 \ldots \mathrm{fib}(n))$ *is the closest integer to $\phi^n/\sqrt{5}$ where $\phi = (1+\sqrt{5})/2$*

Proof. For natural numbers n, the Fibonacci sequence is defined as follows:

$$\mathrm{fib}(n) = \begin{cases} 1 & n = 1, 2 \\ \mathrm{fib}(n-1) + \mathrm{fib}(n-2) & n \geq 3 \end{cases}$$

The proof consists of two parts. First we will show that $\mathrm{fib}(n)$ differs from $\phi^n/\sqrt{5}$ by $\psi^n/\sqrt{5}$ where $\psi = (1-\sqrt{5})/2$. Then we will prove that this difference is strictly less than 0.5 for all natural numbers n. Observe the following two cases:

For n=1:

$$\frac{(\frac{1+\sqrt{5}}{2})^1}{\sqrt{5}} - \frac{(\frac{1-\sqrt{5}}{2})^1}{\sqrt{5}} = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{\sqrt{5}+\sqrt{5}}{2\sqrt{5}} = 1$$

For n=2:

$$\frac{(\frac{1+\sqrt{5}}{2})^2}{\sqrt{5}} - \frac{(\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} = \frac{(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{2\sqrt{5}+2\sqrt{5}}{4\sqrt{5}} = 1$$

So if we set $n = 3$, we know the following two identities hold:

$$\mathrm{fib}(n-1) = \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} \quad \text{and} \quad \mathrm{fib}(n-2) = \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$

We want to show that for all $n \in \mathcal{N}, n \geq 3$, $\mathrm{fib}(n) = fib(n-1) + fib(n-2)$.

$$\begin{aligned} \frac{\phi^n - \psi^n}{\sqrt{5}} &= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\ \phi^2 \cdot \phi^{n-2} - \psi^2 \cdot \psi^{n-2} &= \phi \cdot \phi^{n-2} - \psi \cdot \psi^{n-2} + \phi^{n-2} - \psi^{n-2} \\ \phi^2 \cdot \phi^{n-2} - \psi^2 \cdot \psi^{n-2} &= (\phi+1)\phi^{n-2} - (\psi+1)\psi^{n-2} \qquad (1) \end{aligned}$$

To prove that identity (1) holds, we must equate the coefficients and prove that $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$:

For ϕ:

$$
\begin{aligned}
(\frac{1+\sqrt{5}}{2})^2 &= \frac{1+\sqrt{5}}{2} + 1 \\
\frac{1+2\sqrt{5}+5}{4} &= \frac{1+\sqrt{5}}{2} + \frac{2}{2} \\
\frac{6+2\sqrt{5}}{4} &= \frac{1+\sqrt{5}+2}{2} \\
\frac{3+\sqrt{5}}{2} &= \frac{3+\sqrt{5}}{2}
\end{aligned}
$$

For ψ:

$$
\begin{aligned}
(\frac{1-\sqrt{5}}{2})^2 &= \frac{1-\sqrt{5}}{2} + 1 \\
\frac{1-2\sqrt{5}+5}{4} &= \frac{1-\sqrt{5}}{2} + \frac{2}{2} \\
\frac{6-2\sqrt{5}}{4} &= \frac{1-\sqrt{5}+2}{2} \\
\frac{3-\sqrt{5}}{2} &= \frac{3-\sqrt{5}}{2}
\end{aligned}
$$

So identity (1) holds and we have proven that $\mathrm{fib}(n)$ differs from $\phi^n/\sqrt{5}$ by $\psi^n/\sqrt{5}$.

Now we will show that this difference is less than 0.5. Concretely, this means that for all $n \in \mathcal{N}$, $\left|\phi^n/\sqrt{5}\right| < 0.5$. $\psi \approx -0.618$ so if $n = 1$, we have $\psi/\sqrt{5} \approx -0.276\ldots$, the absolute value of which is less than 0.5. And because $|\psi| < 1$, $\left|\psi^{n+1}\right| < |\psi^n|$ and for all $n \in \mathcal{N}$, $\left|\psi^n/\sqrt{5}\right| < 0.5$.

Therefore, for all natural numbers n, $\mathrm{fib}(n)$ differs from $\phi^n/\sqrt{5}$ by less than 0.5 and the theorem is proved. $\square$